SS1 FURTHER MATHEMATICS LESSON NOTES

FIRST TERM

WEEK 1: SETS (I)

TOPIC: INTRODUCTION TO SETS

INTRODUCTION

In mathematics, **Sets** are the *foundation* of all topics — from algebra and geometry to logic and probability.

Think of a **set** as a *container* or *collection* that holds things (called **elements**) which share a common property.
Illustration 1 — Concept of a Set

```
Set of Fruits (F)
F = {, , , }
Each fruit is a distinct element of the set F.
```

Sets are used everywhere:

- The set of students in a class.
- The set of natural numbers.
- The set of letters in the English alphabet.

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1. DEFINITION OF A SET

A **set** is a *well-defined collection of distinct objects* (called elements).
Keywords:

- **Well-defined:** You can clearly say if something belongs or doesn't belong to the set.
- **Distinct:** No repetition of elements.

✅ **Example (Well-defined)**: A = {The months of the year} → {January, February, …, December}

❌ **Not Well-defined:** B = {Beautiful flowers} → The word *beautiful* is subjective.

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2. SET NOTATION

There are two main ways to describe sets:

A. Roster (Listing) Method

Elements are listed one by one, enclosed in curly brackets **{ }**.

Example: A = {2, 4, 6, 8, 10}

B. Set-builder Notation

We describe the property that defines the elements.

Example: $A = \{x \mid x \text{ is an even number}, 0 < x \leq 10\}$ (Read as: *A is the set of all x such that x is an even number between 0 and 10.*)

COMMON SYMBOLS IN SET THEORY

Symbol	Meaning	Example
$\in$	"is an element of"	$4 \in \{1,2,3,4,5\}$
$\notin$	"is not an element of"	$7 \notin \{1,2,3,4,5\}$
$\subset$	Subset	$A \subset B$
$\subseteq$	Subset (may be equal)	$A \subseteq A$
$\varnothing$	Empty Set	$\varnothing \subset A$
U	Universal Set	All elements considered

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3. TYPES OF SETS

Type	Description	Example
Null Set (∅)	Contains no elements	∅ = { } (humans who can fly unaided)
Singleton Set	Has only one element	{The President of Nigeria}
Finite Set	Countable number of elements	{1, 2, 3, 4, 5}
Infinite Set	Endless elements	{1, 2, 3, 4, …}
Equal Sets	Contain exactly the same elements	{1,2,3} = {3,1,2}
Equivalent Sets	Contain same number of elements	{a,b,c} and {1,2,3}

Illustration 2 — Types of Sets

```
[ ∅ ] → Null Set
[ {A} ] → Singleton
```

Set [{1,2,3}] →
Finite Set
[{1,2,3,4,…}] → Infinite Set

4. UNIVERSAL SET (U)

The **Universal Set (U)** is the set of all objects under discussion.

Example: If U = {1,2,3,4,5,6,7,8,9,10}, and A = {2,4,6,8,10}, then A is a subset of U.

Note: The Universal Set depends on context. For geometry, U may be “all triangles”; for probability, “all possible outcomes”.

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5. SUBSETS (⊂, ⊆)

If every element of A is also in B, then A is a subset of B.

Example: A = {1,2} B = {1,2,3,4} ⇒ A ⊂ B

Every set has:

- **Proper subsets:** Excludes the full set itself.
- **Improper subsets:** Includes the set itself and ∅ .

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6. POWER SET (P(A))

The **power set** of a set A is the *set of all possible subsets of A*.

If n(A) = number of elements in A, then **n(P(A)) = 2^n.**

Example: A = {a, b} Subsets = ∅ , {a}, {b}, {a,b} Hence, P(A) = {∅ , {a}, {b}, {a,b}} and n(P(A)) = 4.

Illustration 3 — Power Set Tree

```
      {a,b}
     /     \
  {a}       {b}
    \       /
       ∅
```

WORKED EXAMPLES

Example 1: Let V = {a, e, i, o, u} a) Write in set-builder form. b) Is it finite or infinite? c) Find n(V). d) Is a ∈ V? e) Is b ∈ V?

✅ **Solution:** a) V = {x | x is a vowel in English alphabet} b) Finite (since we can count all 5 vowels) c) $n(V) = 5$ d) Yes, $a \in V$ e) No, $b \notin V$

Example 2: Let C = {red, green} a) List all subsets. b) Write the power set P(C). c) Find n(P(C)).

✅ **Solution:** Subsets = ∅ , {red}, {green}, {red, green} P(C) = {∅ , {red}, {green}, {red, green}} n(P(C)) = 4

REVISION QUESTIONS

1. Which of the following are well-defined? a) The set of interesting subjects in SS1. b) The set of rivers in Nigeria. c) The set of tasty meals.
2. List all elements of P = {x | x is a prime number and 10 < x < 25}.
3. Let A = {1, {2,3}, 4}. Is {2,3} ∈ A? Is 2 ∈ A?
4. If n(X) = 5, find the number of subsets and proper subsets of X.

ASSIGNMENT (TAKE-HOME)

1. Classify as finite or infinite: a) All leaves on trees in Lagos. b) A = {x | x is an integer < 0}. c) B = {people who are 300 years old}.
2. Let U = {x | 0 < x ⩽ 12, x ∈ N}, A = {factors of 12}. a) List elements of U. b) List elements of A. c) Is A ⊂ U?
3. If n(P) = k, find n(P(P)).
4. A set X has 63 proper subsets. Find the number of elements in X.
5. Given S = {a,b,c,d,e}, list all subsets with exactly 3 elements.

WEEK 2: SETS (II)

TOPIC: SET OPERATIONS AND VENN DIAGRAMS

INTRODUCTION

Last week, we described single sets. Now, we'll learn how **sets interact** — combining, overlapping, and excluding elements. The visual tool for this is the **Venn Diagram**, invented by **John Venn (1880)**.

Illustration 4 — Basic 2-Set Venn Diagram

```
┌──────────── U ────────────┐
│         ○ A       ○ B     │
│       (Overlap = A ∩ B)   │
└───────────────────────────┘
```

1. SET OPERATIONS

Operation	Symbol	Definition	Example
Union	$\cup$	Elements in A or B or both	$A=\{1,2,3\}$, $B=\{3,4,5\}$, $A\cup B=\{1,2,3,4,5\}$
Intersection	$\cap$	Elements common to both	$A\cap B=\{3\}$
Complement	A'	Elements in U but not in A	If $U=\{1–5\}$, $A=\{1,2,3\}$, $A'=\{4,5\}$
Difference	$A\setminus B$	Elements in A but not in B	$A\setminus B=\{1,2\}$

Illustration 5 — Union and Intersection

```
Union (A∪B): Both circles shaded
Intersection (A∩B): Only the overlap
shaded
```

2. PRINCIPLE OF INCLUSION–EXCLUSION

For two sets:

$$n(A\cup B) = n(A) + n(B) - n(A\cap B)$$

For three sets:

$$n(A\cup B\cup C) = n(A) + n(B) + n(C) - n(A\cap B) - n(A\cap C) - n(B\cap C) + n(A\cap B\cap C)$$

3. VENN DIAGRAMS

A **Venn Diagram** shows sets pictorially.

- The rectangle = Universal Set (U).
- Circles = Subsets (A, B, C).
- Overlaps = Intersections.

Illustration 6 — 3-Set Venn Diagram Layout

```
A ∩ B ∩ C
(center) A ∩ B
only
A only
B only
C only
Outside region = (A∪B∪C)'
```

---

4. WORKED EXAMPLES

Example 1 — 2-Set Problem

In a class of 35 students, 20 offer Physics (P) and 17 offer History (H). 5 offer neither.

Find: a) Students who offer both. b) Students who offer History only.

✅ **Solution:** $n(U)=35$, $n(P)=20$, $n(H)=17$, "Neither"$=5$

$n(P\cup H)=35-5=30$ Use inclusion-exclusion: $n(P\cup H)=n(P)+n(H)-n(P\cap H)$ $30=20+17-n(P\cap H)$ ~ $n(P\cap H)=7$

a) **Both = 7** b) **History only = $n(H)-n(P\cap H)=17-7=10$**

Illustration 7 — 2-Set Diagram

```
   (A∩B=7)
A=20   total   →   A
only=13 B=17 total →
B only=10 Outside=5
```

Example 2 — 3-Set Problem

Data	Value

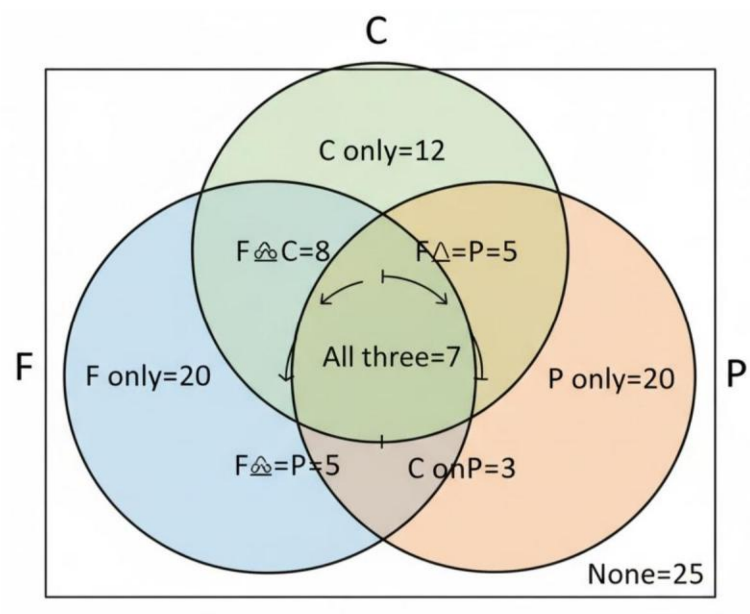
n(U)	100
n(F)	40
n(C)	30
n(P)	35

Data	Value
n(F∩C)	15
n(F∩P)	12
n(C∩P)	10
n(F∩C∩P)	7

✅ Solution:

- n(F∩C only)=15−7=8
- n(F∩P only)=12−7=5
- n(C∩P only)=10−7=3
- n(F only)=40−(8+5+7)=20
- n(C only)=30−(8+3+7)=12
- n(P only)=35−(5+3+7)=20

Total inside = 75 “None” = 100−75=25
Illustration 8 — 3-Set Venn Diagram



5. DE MORGAN'S LAWS

For any sets A and B:

1. $(A \cup B)' = A' \cap B'$
2. $(A \cap B)' = A' \cup B'$

Illustration: Overlapping circles showing how complements flip union/intersection.

REVISION QUESTIONS

1. If U={1,2,…,10}, A={1,3,5,7,9}, B={2,3,5,7}, find: a) $A\cup B$ b) $A\cap B$ c) A' d) $B\setminus A$
2. Shade region $B\cap A'$ in a Venn diagram.
3. State De Morgan's Laws.
4. If 15 play Football, 12 play Basketball, and 5 both, how many students are there if all play at least one?

ASSIGNMENT (TAKE-HOME)

1. Given U={a,b,c,d,e,f,g,h}, X={a,b,d,e}, Y={b,e,f,g}: Find: a) $X'\cap Y$ b) $(X\cup Y)'$ c) Verify $(X\cup Y)'=X'\cap Y'$
2. In a group of 50 people, 30 speak English, 25 speak French, and 10 speak neither: a) Draw a Venn diagram. b) Find number who speak both. c) Find English only.
3. A survey of 60 fans: 28 watch Football (F), 22 Basketball (B), 25 Cricket (C), 10 $F\cap B$, 8 $F\cap C$, 7 $B\cap C$, 3 all three. a) Fill all 8 regions. b) How many watch exactly two? c) How many watch none?
4. If $n(A\setminus B)=15$, $n(B\setminus A)=10$, $n(A\cap B)=5$, find $n(A\cup B)$.
5. Draw and shade region $A\cap(B\cup C)'$.

WEEK 3: INDICES AND LOGARITHMS

TOPIC: INDICES AND LOGARITHMS

INTRODUCTION

Indices and logarithms are two powerful tools that help us express very large or very small numbers and solve exponential equations efficiently.

- **Indices (or powers)** express repeated multiplication compactly. Example: (2^5 = 2 2 2 2 2 = 32)
- **Logarithms** do the reverse of indices. If $10^2 = 100$, then $\log_{10}100 = 2$. This means "10 must be raised to the power of 2 to get 100."

Understanding both is crucial for later work in **exponential growth, compound interest, and calculus**.

SECTION A: INDICES

1. LAWS OF INDICES

For any positive base a and integers m, n:

Law	Algebraic Form	Example
Multiplication	$a^m \times a^n = a^{m+n}$	$2^3 \times 2^4 = 2^7 = 128$
Division	$a^m \div a^n = a^{m-n}$	$5^6 \div 5^2 = 5^4 = 625$
Power	$(a^m)^n = a^{mn}$	$(3^2)^4 = 3^8 = 6561$
Zero Index	$a^0 = 1$	$7^0 = 1$
Negative Index	$a^{-n} = 1/a^n$	$2^{-3} = 1/8$
Fractional Index	$a^{1/n} = \sqrt[n]{a}$	$8^{1/3} = \sqrt[3]{8} = 2$
Mixed Fractional	$a^{m/n} = (\sqrt[n]{a})^m$	$27^{2/3} = (^3\sqrt{27})^2 = 9$

ILLUSTRATION 1 – The Meaning of Indices

```
a^5 = a × a × a × a × a
```

Exponent (5) → tells how many times the **base (a)** is multiplied by itself.

```
     a × a × a × a × a
     ↑               ↑
   base            power (exponent)
```

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2. APPLICATIONS OF INDICES

Example 1: Simplify $(16/81)^{-3/4}$

Solution:

$$(16/81)^{-3/4} = (81/16)^{3/4}$$

$$= \left(\frac{\sqrt[4]{81}}{\sqrt[4]{16}}\right)^3 = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$$

Example 2: Solve $4^x - 6(2^x) + 8 = 0$

Let $y = 2^x$ ⇒ $4^x = (2^2)^x = (2^x)^2 = y^2$.

Then:

$$y^2 - 6y + 8 = 0$$

$$(y-2)(y-4) = 0$$

So, $y = 2$ or $y = 4$

Substitute back $2^x = 2 \Rightarrow x = 1$;

$2^x = 4 \Rightarrow x = 2$

✅ **Final Answer:** $x = 1 \text{ or } 2$

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3. GRAPHICAL INTERPRETATION OF EXPONENTIAL GROWTH

Function	Behavior
$y = 2^x$	Increases rapidly as x increases
$y = (1/2)^x$	Decreases (decay curve)

(Imagine two curves on a Cartesian plane: one rising steeply, one decaying.)

SECTION B: LOGARITHMS

1. DEFINITION

If $a^x = N$, then $\log_a N = x$

> **Read as:** "logarithm of N to base a equals x."

Example:

$10^3 = 1000 \Rightarrow \log_{10} 1000 = 3$

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2. LAWS OF LOGARITHMS

Law	Algebraic Form	Example
Product	$\log_a(MN) = \log_a M + \log_a N$	$\log_{10}(5 \times 4) = \log_{10} 5 + \log_{10} 4$
Quotient	$\log_a(M/N) = \log_a M - \log_a N$	$\log_{10}(50/2) = \log_{10} 50 - \log_{10} 2$
Power	$\log_a(M^p) = p \log_a M$	$\log_{10} 8^2 = 2 \log_{10} 8$

Corollaries:

- $\log_a a = 1$
- $\log_a 1 = 0$

---

3. CHANGE OF BASE

$$\log_a b = \frac{\log_c b}{\log_c a}$$

Usually, $c = 10$ or e (for natural logs).

Corollary: $\log_a b = \frac{1}{\log_b a}$

ILLUSTRATION 2 – Relationship Between Indices and Logarithms

If $a^x = N$, then $\log_a N = x$.

$a^x = N \quad \leftrightarrow \quad \log_a N = x$

Example:

$2^5 = 32 \leftrightarrow \log_2 32 = 5$

WORKED EXAMPLE (Logarithm Equation)

Example 3: Solve for x if $\log_2 x + \log_2(x - 7) = 3$

Solution:

$$\log_2[x(x-7)] = 3$$

$$x^2 - 7x = 8$$

$$x^2 - 7x - 8 = 0$$

$$(x-8)(x+1) = 0$$

Valid value: $x = 8$ (since $x > 7$) ✅

WORKED EXAMPLE (Change of Base)

Given $\log_3 2 = 0.631$, find $\log_2 9$.

$$\log_2 9 = \frac{\log_3 9}{\log_3 2}$$

$$\log_3 9 = 2 \Rightarrow \frac{2}{0.631} = 3.17$$

✅ **Answer:** $\log_2 9 = 3.17$

EVALUATION QUESTIONS

1. Simplify $(x^{1/2}y^2)^{1/3} \div (x^{-1/3}y)^{1/2}$
2. Solve $(1/4)^x = 32$
3. Express as a single logarithm: $\frac{1}{2}\log 25 - 2\log 3 + \log 18$
4. Given $\log_{10} 2 = 0.3010$, find $\log_{10} 32$
5. Solve $5^{2k} + 2(5^k) - 3 = 0$

ASSIGNMENT

1. Simplify $((125)^{1/3} \times (27)^{2/3})/((64)^{-1/3} \times (81)^{1/4})$
2. Solve $3^{2x} - 10(3^x) + 9 = 0$
3. $\log_3 p - \log_3(p-6) = 2$. Find p.
4. $\log_4(x+1) + \log_4(x-1) = \log_4 8$. Find x.
5. If $\log_2 x = a$ and $\log_2 y = b$, express $\log_2((x^3y)/\sqrt{8})$ in terms of a, b.

WEEK 4: SURDS

TOPIC: SURDS

INTRODUCTION

A **surd** is an irrational number that cannot be simplified to remove the square (or cube) root. For instance, $\sqrt{2}$, $\sqrt{3}$, and $\sqrt{5}$ cannot be expressed as exact fractions or terminating decimals.

Perfect Squares	Surds
$\sqrt{4} = 2$	$\sqrt{2}$ (irrational)
$\sqrt{9} = 3$	$\sqrt{3}$ (irrational)

Thus, a **surd** remains in root form to preserve **accuracy**.

1. TYPES OF SURDS

- **Simple Surd:** $\sqrt{7}$, $\sqrt[3]{5}$
- **Compound Surd:** $2 + \sqrt{3}$, $5 - 2\sqrt{2}$

2. RULES FOR MANIPULATING SURDS

Operation	Rule	Example
Addition/Subtraction	Only like surds combine	$3\sqrt{5} + 2\sqrt{5} = 5\sqrt{5}$
Multiplication	$\sqrt{a} \times \sqrt{b} = \sqrt{(ab)}$	$\sqrt{2} \times \sqrt{3} = \sqrt{6}$
Division	$\sqrt{a} / \sqrt{b} = \sqrt{(a/b)}$	$\sqrt{12} / \sqrt{3} = \sqrt{4} = 2$
Simplification	$\sqrt{(a^2b)} = a\sqrt{b}$	$\sqrt{72} = \sqrt{(36\times2)} = 6\sqrt{2}$

ILLUSTRATION 3 – Simplifying Surds

```
√72 = √(36 × 2) = 6√2


     ________
   √ 36 × 2
     ↑    ↑
 Perfect  Left-over
 Square     factor
```

3. RATIONALIZATION

Objective: Eliminate the surd from the denominator.

Case 1: Monomial Denominator

$$\frac{6}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{6\sqrt{3}}{3} = 2\sqrt{3}$$

Case 2: Binomial Denominator

$$\frac{10}{3-\sqrt{2}} \times \frac{3+\sqrt{2}}{3+\sqrt{2}} = \frac{30+10\sqrt{2}}{7}$$

Concept: Multiply by the *conjugate* to remove the surd.

WORKED EXAMPLES

Example 1: *Simplify √50 + 3√8 − √18*

$$\sqrt{50} = 5\sqrt{2}, \quad \sqrt{8} = 2\sqrt{2}, \quad \sqrt{18} = 3\sqrt{2}$$

$$5\sqrt{2} + 6\sqrt{2} - 3\sqrt{2} = 8\sqrt{2}$$

✅ **Answer: 8√2**

Example 3: Rationalize $\frac{\sqrt{5}}{2+\sqrt{5}}$

Multiply by the conjugate (2 - √5):

$$\frac{\sqrt{5}(2-\sqrt{5})}{(2+\sqrt{5})(2-\sqrt{5})} = \frac{2\sqrt{5}-5}{4-5} = \frac{2\sqrt{5}-5}{-1} = 5-2\sqrt{5}$$

✅ **Answer: 5 – 2√5**

Example 3: Rationalize $\frac{\sqrt{5}}{2+\sqrt{5}}$

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✅ **Answer: 5 – 2√5**

ILLUSTRATION 4 – Rationalization Process

```
            Multiply by conjugate
        ___________________________
       |     1 / (2 + √3) × (2 - √3)/(2 - √3)
       |      = (2 - √3) / (4 - 3)
       ↓
         Result: 2 - √3
```

EVALUATION QUESTIONS

1. Define a surd.
2. Simplify √72.
3. Simplify 4√5 + √20 – √45.
4. Rationalize $\frac{1}{2\sqrt{5}}$.
5. What is the conjugate of $5 - 3\sqrt{2}$?

ASSIGNMENT

1. Simplify √48 + √108 – √27.
2. Expand (√6 + 2)(2√6 – 3).
3. Rationalize $\frac{12}{\sqrt{11}-\sqrt{7}}$.
4. Express $\frac{3\sqrt{2}-1}{2\sqrt{2}+3}$ in the form $p + q\sqrt{2}$.
5. Given $x = 1 + \sqrt{2}$, find $x^2 + \frac{1}{x^2}$.

WEEK 5: THE STRAIGHT LINE IN COORDINATE GEOMETRY

INTRODUCTION

Coordinate Geometry — also called Analytical Geometry — forms the powerful link between Algebra and Geometry. In this branch of mathematics, points, lines, and shapes are described using algebraic equations on a Cartesian plane, allowing us to study geometric figures numerically.

The straight line is the simplest and most fundamental of all geometric forms. Understanding its properties — midpoint, distance, gradient, and equation — is essential for solving higher problems in geometry, trigonometry, and calculus.

CONTENT

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1. MIDPOINT OF A LINE SEGMENT

Concept:

The midpoint of a line segment joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is the point exactly halfway between them.

Formula:

$M((x_1 + x_2)/2, (y_1 + y_2)/2)$

Illustration:

```
A(x₁, y₁) ●──────────────● M ──────────────● B(x₂, y₂)
                    ↑
                 Midpoint
```

The midpoint simply represents the average of the x-coordinates and the average of the y-coordinates.

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2. GRADIENT (SLOPE) OF A STRAIGHT LINE

Concept:

The gradient (m) of a line measures its steepness and direction. It represents the ratio of the vertical change (Rise) to the horizontal change (Run) between any two points on the line.

Formula:

$m = \Delta y/\Delta x = (y_2 - y_1)/(x_2 - x_1)$

Interpretation:

- A positive gradient ($m > 0$) → line rises from left to right.
- A negative gradient ($m < 0$) → line falls from left to right.
- $m = 0$ → horizontal line.
- Undefined m → vertical line.

Diagram:

```
      (x₂, y₂) •
                 \
                  \
                   \
                    • (x₁, y₁)
   Rise = y₂ - y₁
   Run  = x₂ - x₁
```

---

3. DISTANCE BETWEEN TWO POINTS

Concept:

The distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ is derived from Pythagoras' Theorem.

Formula:

$d = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2]}$

Illustration:

```
           B(x₂, y₂)
           •
           |\
           | \
     Δy    |  \  d (distance)
           |   \
           |____\
      A(x₁, y₁)   Δx
```

Here, $\Delta x = x_2 - x_1$ and $\Delta y = y_2 - y_1$.

---

4. CONDITIONS FOR PARALLELISM AND PERPENDICULARITY

A. Parallel Lines

Two lines are parallel if and only if their gradients are equal.

$L_1 \parallel L_2 \Rightarrow m_1 = m_2$

Parallel lines never meet.

B. Perpendicular Lines

Two lines are perpendicular if and only if the product of their gradients is -1.

$L_1 \perp L_2 \Rightarrow m_1 \times m_2 = -1$

or equivalently,

$m_2 = -1/m_1$

Illustration:

```
Parallel Lines:      Perpendicular Lines:
     ↗                    ↗
    ↗  ↗                 ↘ ↗
   ↗    ↗               ↘    ↗
```

---

5. EQUATION OF A STRAIGHT LINE

A. Gradient–Intercept Form

$y = mx + c$

where m = gradient, c = y-intercept (the point where the line cuts the y-axis).

Example: $y = 2x + 3$ means the line crosses the y-axis at (0, 3) and has gradient 2.

B. Point–Gradient Form

$y - y_1 = m(x - x_1)$

Used when the gradient and one point (x_1, y_1) are known.

C. General Form

$Ax + By + C = 0$

All terms are on one side of the equation.

Example: $2x - y - 7 = 0$ can be rewritten as $y = 2x - 7$.

---

6. TRANSFORMING RELATIONS INTO LINEAR FORM

Purpose:

In experiments, some non-linear relationships (e.g., $y = ax^n$ or $y = ab^x$) can be linearized using logarithms. When converted to straight-line form, constants such as a, b, and n can be found from graphs.

Case 1: $y = ax^n$

Take logs of both sides:

$\log y = \log a + n \log x$

Comparing with $Y = mX + C$:

$Y = \log y$, $X = \log x$, $m = n$, $C = \log a$

Thus, a plot of log y (vertical axis) against log x (horizontal axis) gives a straight line with:

- Slope = n
- Intercept = log a

Case 2: $y = ab^x$

Take logs of both sides:

$\log y = \log a + x \log b$

Comparing with $Y = mX + C$:

$Y = \log y$, $X = x$, $m = \log b$, $C = \log a$

Thus, a plot of log y (Y-axis) against x (X-axis) gives a straight line.

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7. AREA OF TRIANGLES AND QUADRILATERALS (SHOELACE METHOD)

Concept:

The area of a polygon whose coordinates are known can be found using the Shoelace Formula (also called Determinant Formula).

For a Triangle:

Given points $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$,

$$\text{Area} = \tfrac{1}{2} \left|(x_1y_2 + x_2y_3 + x_3y_1) - (y_1x_2 + y_2x_3 + y_3x_1)\right|$$

Diagrammatic Representation:

```
Vertices connected in order:
A(x₁,y₁)
   \
    \
     •C(x₃,y₃)
     /
    /
B(x₂,y₂)
```

For a Quadrilateral:

Divide the shape into two triangles, calculate their individual areas, and add them together.

ROBUST WORKED EXAMPLES

Example 1: Midpoint, Distance, and Gradient

Given: P(-2,5) and Q(4,1)

Find: (a) Midpoint (M), (b) Distance (d), (c) Gradient (m)

Solution:

Let $(x_1, y_1) = (-2,5)$, $(x_2, y_2) = (4,1)$

(a) $M = ((-2 + 4)/2, (5 + 1)/2) = (1, 3)$

(b) $d = \sqrt{[(4 - (-2))^2 + (1 - 5)^2]} = \sqrt{[6^2 + (-4)^2]} = \sqrt{52} = 2\sqrt{13}$

(c) $m = (1 - 5)/(4 - (-2)) = -4/6 = -2/3$

Example 2: Parallel and Perpendicular Lines

Find the equation of a line passing through (3, -1) and perpendicular to $2y + x = 5$.

Solution:

$2y + x = 5 \Rightarrow y = -\tfrac{1}{2}x + 5/2$

Gradient of given line $m_1 = -\tfrac{1}{2}$

Perpendicular line $\rightarrow m_2 = -1/m_1 = 2$

Use point-gradient form:

$y - (-1) = 2(x - 3)$

$y + 1 = 2x - 6$

$y = 2x - 7$ or $2x - y - 7 = 0$

Example 3: Area of a Triangle (Shoelace Formula)

Given: A(1,1), B(4,5), C(6,2)

Area = $\frac{1}{2} |(1\times5 + 4\times2 + 6\times1) - (1\times4 + 5\times6 + 2\times1)|$

$= \frac{1}{2} |(5 + 8 + 6) - (4 + 30 + 2)|$

$= \frac{1}{2} |19 - 36| = \frac{1}{2} \times 17 = 8.5$

Answer: Area = 8.5 square units

EVALUATION (IN-CLASS)

1. Find the distance between A(1, –2) and B(4, 2).
2. Find the midpoint of the line segment joining P(–3, 0) and Q(5, –6).
3. Find the gradient of the line passing through (1, 7) and (3, 1).
4. A line has the equation $3x - y = 6$. Find its gradient and y-intercept.
5. Are the lines $y = 2x + 1$ and $4x - 2y + 5 = 0$ parallel? Justify your answer.

ASSIGNMENT (TAKE-HOME)

1. A line L_1 passes through (2,5) and (4,9). A line L_2 passes through (0,1) and (–4,3). Determine if L_1 and L_2 are parallel, perpendicular, or neither.
2. Find the equation of the line parallel to $y = 3x - 1$ that passes through (2, 0).
3. Find the equation of the perpendicular bisector of the line joining A(–1,4) and B(3,2).
4. The variables x and y are related by $y = kx^n$. The table shows:

x	1	4	9
y	5	10	15

Explain how to plot a straight-line graph to find k and n. (What would you plot on the X and Y axes?)

5. Find the area of the quadrilateral with vertices P(1,2), Q(3,5), R(6,6), S(4,1).

WEEK 6: TRIGONOMETRIC RATIOS OF SPECIAL ANGLES & LOGICAL REASONING (PART I)

TOPIC 1: TRIGONOMETRIC RATIOS OF 30°, 45°, AND 60°

INTRODUCTION

Trigonometry is a branch of mathematics that studies the relationships between the sides and angles of triangles. The three basic trigonometric ratios are:

- **Sine (sin) = Opposite / Hypotenuse**
- **Cosine (cos) = Adjacent / Hypotenuse**
- **Tangent (tan) = Opposite / Adjacent**

Among all possible angles, 30°, 45°, and 60° are called special angles because their trigonometric ratios have exact values that can be derived geometrically — without using a calculator.

These angles frequently appear in geometry, physics, navigation, and engineering, and understanding them gives a strong foundation for solving trigonometric problems.

CONTENT

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1. DERIVATION OF TRIGONOMETRIC RATIOS FOR 45°

We begin with an isosceles right-angled triangle, where the two non-right angles are equal — each 45°.

Let each of the equal sides be 1 unit.

Diagram:

```
          /|
         / |
   1    /  |  1
       /   |
      /____|
       45° 45°
```

Step 1: Use Pythagoras' Theorem to find the hypotenuse

$h^2 = 1^2 + 1^2 = 2 \Rightarrow h = \sqrt{2}$

Step 2: Apply SOH–CAH–TOA

- sin 45° = Opposite / Hypotenuse = $1 / \sqrt{2} = \sqrt{2} / 2$
- cos 45° = Adjacent / Hypotenuse = $1 / \sqrt{2} = \sqrt{2} / 2$
- tan 45° = Opposite / Adjacent = 1 / 1 = 1

✅ Therefore:

Ratio	Exact Value
sin 45°	$\sqrt{2}/2$
cos 45°	$\sqrt{2}/2$
tan 45°	1

---

2. DERIVATION OF TRIGONOMETRIC RATIOS FOR 30° AND 60°

Now, we derive from an equilateral triangle (all sides and all angles equal).

Let each side = 2 units, and each angle = 60°.

Step 1: Draw and bisect the triangle vertically

```
        /\
       /|\
      / | \
     /  |  \
    /___|___\
     1  h   1
    30°     60°
```

The bisector divides the 60° vertex into two 30° angles and splits the base (2 units) into 1 unit each.

Step 2: Find the height (h) using Pythagoras' theorem

$2^2 = 1^2 + h^2 \Rightarrow 4 = 1 + h^2 \Rightarrow h^2 = 3 \Rightarrow h = \sqrt{3}$

Step 3: Apply SOH–CAH–TOA

For the 60° angle:

- $\sin 60° = \text{Opposite} / \text{Hypotenuse} = \sqrt{3} / 2$
- $\cos 60° = \text{Adjacent} / \text{Hypotenuse} = 1 / 2$
- $\tan 60° = \text{Opposite} / \text{Adjacent} = \sqrt{3} / 1 = \sqrt{3}$

For the 30° angle:

- $\sin 30° = \text{Opposite} / \text{Hypotenuse} = 1 / 2$
- $\cos 30° = \text{Adjacent} / \text{Hypotenuse} = \sqrt{3} / 2$
- $\tan 30° = \text{Opposite} / \text{Adjacent} = 1 / \sqrt{3} = \sqrt{3} / 3$

3. SUMMARY TABLE OF SPECIAL ANGLES

Angle (θ)	sin θ	cos θ	tan θ
30°	1/2	$\sqrt{3}/2$	$\sqrt{3}/3$ (or $1/\sqrt{3}$)
45°	$\sqrt{2}/2$	$\sqrt{2}/2$	1
60°	$\sqrt{3}/2$	1/2	$\sqrt{3}$

4. RECIPROCAL TRIGONOMETRIC RATIOS

Every trigonometric ratio has a reciprocal (or inverse) ratio:

Function	Reciprocal Function	Definition	Example
sin θ	csc θ	$\csc\theta = 1/\sin\theta$	$\csc 30° = 1/(½) = 2$
cos θ	sec θ	$\sec\theta = 1/\cos\theta$	$\sec 60° = 1/(½) = 2$
tan θ	cot θ	$\cot\theta = 1/\tan\theta$	$\cot 45° = 1/1 = 1$

ROBUST WORKED EXAMPLES

Example 1: Evaluating Expressions

Find the exact value of $(\sin 60° + \cos 60°) / \tan 45°$

Solution:

Substitute the values:

$(\sqrt{3}/2 + 1/2) / 1$

Simplify numerator:

$(\sqrt{3} + 1)/2$

✅ **Answer:** $(\sqrt{3} + 1) / 2$

Example 2: Solving for an Unknown

Find x if $x \sin 30° \cos 45° = \tan 60°$.

Solution:

Substitute known values:

$x (½)(\sqrt{2}/2) = \sqrt{3}$

Simplify:

$x (\sqrt{2}/4) = \sqrt{3}$

Multiply both sides by $(4/\sqrt{2})$:

$x = (4\sqrt{3})/\sqrt{2}$

Rationalize denominator:

$x = (4\sqrt{3}\times\sqrt{2})/2 = 2\sqrt{6}$

✅ **Answer:** $x = 2\sqrt{6}$

Example 3: Application in Geometry

In ΔABC, ∠B = 90°, ∠A = 30°, and AC = 10 cm. Find the exact lengths of AB and BC.

Solution:

We know:

- sin 30° = Opp/Hyp = BC / 10
- cos 30° = Adj/Hyp = AB / 10
- sin 30° = ½ ⇒ ½ = BC / 10 ⇒ BC = 5 cm
- cos 30° = $\sqrt{3}/2$ ⇒ $\sqrt{3}/2$ = AB / 10 ⇒ AB = $5\sqrt{3}$ cm

✅ **Answer:** BC = 5 cm, AB = $5\sqrt{3}$ cm

Example 4: Using Reciprocal Ratios

Find sec 30° × csc 60°.

Solution:

- $\sec 30° = 1 / \cos 30° = 1 / (\sqrt{3}/2) = 2 / \sqrt{3}$
- $\csc 60° = 1 / \sin 60° = 1 / (\sqrt{3}/2) = 2 / \sqrt{3}$

Multiply:

$(2 / \sqrt{3}) \times (2 / \sqrt{3}) = 4 / 3$

✅ **Answer:** 4/3

VISUAL RELATIONSHIPS OF SPECIAL ANGLES

```
sin 30° = 1/2
cos 30° = √3/2

      /|
     / |
    /  | √3
   /θ  |
  /____|
   1
(Hypotenuse = 2)
```

KEY CONCEPTS TO REMEMBER

- The sum of squares identity always holds: $\sin^2 \theta + \cos^2 \theta = 1$
- $\tan \theta = \sin \theta / \cos \theta$
- The reciprocal identities: $\csc \theta = 1/\sin \theta$, $\sec \theta = 1/\cos \theta$, $\cot \theta = 1/\tan \theta$

EVALUATION (IN-CLASS)

1. Draw a 30°–60°–90° triangle and derive sin 60° and tan 30°.
2. Find the exact value of sin 45° + cos 45°.
3. Compute $(\tan 60°)^2 + (\sin 30°)^2$.
4. Find the value of sec 30°.
5. If $2y \cos 30° = \tan 60°$, find y exactly.

ASSIGNMENT (TAKE-HOME)

1. Evaluate: $(\sin 60° \cos 30°) / (\tan 45° – \sin 30°)$
2. A ladder leans against a wall at 60°. If the base is 4 m from the wall, find the exact length of the ladder. (Hint: Use cos 60° = Adj/Hyp)
3. Solve for θ (acute): $2 \cos \theta = 1$

4. Find the exact value of $1 + \tan^2 60°$ (show that it equals $\sec^2 60°$).
5. Without using a calculator, verify: $\sin^2 30° + \cos^2 30° = 1$

CONCLUSION

Understanding these special angles forms the foundation of trigonometry. By mastering their geometric derivations and exact values, you can:

- Solve triangles without calculators
- Simplify trigonometric equations
- Apply trigonometry in physics, architecture, and navigation

Tip: Memorize the special angle table (30°, 45°, 60°). It appears in almost every trigonometric problem in both school exams and external tests like WAEC, NECO, and UTME.

WEEK 7: TRIGONOMETRIC RATIOS & LOGICAL REASONING (PART II)

TOPIC: INTRODUCTION TO LOGICAL REASONING

INTRODUCTION

Logical reasoning is the backbone of mathematical thinking. It allows us to analyze arguments, evaluate truth, and form valid conclusions. In mathematics, we often deal with statements (called propositions) that can be either true (T) or false (F).

Logical reasoning helps us understand how the truth of one statement affects another, and how we can build compound statements using logical connectives like and ($\wedge$), or ($\vee$), not (~), if...then (→), and if and only if (↔).

This concept is not only useful in mathematics but also in computer programming, engineering, and critical thinking in everyday life.

CONTENT

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1. PROPOSITIONS (STATEMENTS)

A proposition (or statement) is a declarative sentence that can be clearly classified as either True (T) or False (F), but not both.

Example	Statement	Type	Truth Value
p	Lagos is the capital of Nigeria.	Proposition	False
q	$2 + 2 = 4$	Proposition	True
r	What is your name?	Not a proposition	—
s	Go and shut the door.	Not a proposition	—
t	$x + 1 = 5$	Open sentence (depends on x)	—

A proposition must be definite — its truth value should not depend on unknown variables.

Visual Representation

```
┌──────────────────────────────────────┐
│  SENTENCES IN LANGUAGE               │
├──────────────────────────────────────┤
│  1. Questions → Not Propositions     │
│  2. Commands → Not Propositions      │
│  3. Declarative → Propositions       │
└──────────────────────────────────────┘
```

2. LOGICAL CONNECTIVES AND THEIR TRUTH TABLES

Logical connectives are symbols that join simple propositions to form compound propositions.

We'll explore five main logical connectives and their corresponding truth tables.

A. NEGATION (~p or ¬p)

This simply reverses the truth value of a proposition. If p is True, then ~p is False, and vice versa.

p	~p
T	F
F	T

Example: p: "It is sunny." → ~p: "It is not sunny."

B. CONJUNCTION (p ∧ q)

The conjunction "p and q" is True only when both p and q are True.

p	q	p ∧ q
T	T	T
T	F	F
F	T	F
F	F	F

Example: p: "I passed the test." q: "I studied hard." p ∧ q means "I passed the test and I studied hard."

C. DISJUNCTION (p ∨ q)

This is the inclusive "or", meaning p ∨ q is True if either p or q is True, or both.

p	q	p ∨ q
T	T	T
T	F	T
F	T	T
F	F	F

Example: p: "I will travel to Lagos." q: "I will travel to Abuja." p ∨ q means "I will travel to Lagos or Abuja (or both)."

D. CONDITIONAL (p → q)

This represents "if p, then q." It is False only when p is True and q is False.

p	q	p → q
T	T	T
T	F	F
F	T	T
F	F	T

Explanation (with analogy): "If I get an A (p), I will buy you ice cream (q)." I only lie (the statement is False) if I get an A but do not buy you ice cream.

E. BICONDITIONAL (p ↔ q)

This represents "p if and only if q." It is True only when p and q have the same truth value.

p	q	p ↔ q
T	T	T
T	F	F
F	T	F
F	F	T

Example: p: "Two angles are equal." q: "The triangle is isosceles." p ↔ q means the triangle is isosceles if and only if two angles are equal.

---

3. TAUTOLOGY, CONTRADICTION, AND CONTINGENCY

Type	Definition	Example	Truth Value
Tautology	Always True, no matter what p or q are.	p ∨ ~p	Always T

Type	Definition	Example	Truth Value
Contradiction	Always False, no matter what p or q are.	$p \land \sim p$	Always F
Contingency	Sometimes True, sometimes False.	$p \to q$	Depends on p and q

Diagrammatic View

```
┌─────────────────────────────────┐
│ LOGICAL STATEMENTS              │
├─────────────────────────────────┤
│ • Tautology → Always True       │
│ • Contradiction → Always False  │
│ • Contingency → Sometimes True  │
└─────────────────────────────────┘
```

ROBUST WORKED EXAMPLES

Example 1: Translating to Symbols

Let p = "It is cold" q = "It is raining"

Translate:

a) It is cold and it is raining → $p \land q$ b) It is not cold or it is raining → $\sim p \lor q$ c) If it is raining, then it is not cold → $q \to \sim p$ d) It is neither cold nor raining → $\sim p \land \sim q$

Example 2: Constructing a Truth Table

Construct a truth table for: $(p \to q) \land \sim q$

p	q	~q	p → q	(p → q) ∧ ~q
T	T	F	T	F
T	F	T	F	F
F	T	F	T	F
F	F	T	T	T

✅ Final column is not constant, so it's a contingency.

Example 3: Checking for Tautology

Is $(p \land q) \to p$ a tautology?

p	q	p ∧ q	(p ∧ q) → p

p	q	p ∧ q	(p ∧ q) → p
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

✅ The last column is always True → Tautology

Example 4: Identifying a Contradiction

Check if (p ∧ ~p) is a contradiction.

p	~p	p ∧ ~p
T	F	F
F	T	F

✅ Always False → Contradiction

Example 5: Logical Reasoning in Real Life

Statement: "If a number is even, then it is divisible by 2."

Let p: "A number is even." Let q: "It is divisible by 2." → This can be written as p → q

Truth Table confirms it's a tautology, because it's always true by definition.

VISUAL SUMMARY

LOGICAL CONNECTIVES MAP

```
┌──────────────────────────────────────────────┐
│ Simple Statements (p, q)                      │
│         │                                     │
│         ├── Negation (~p) → reverses truth    │
│         ├── Conjunction (p ∧ q) → both true   │
│         ├── Disjunction (p ∨ q) → either true │
│         ├── Conditional (p → q) → if...then   │
│         └── Biconditional (p ↔ q) → both same │
└──────────────────────────────────────────────┘
```

EVALUATION (IN-CLASS)

1. Which of the following are propositions? a) $x > 5$ b) $10 - 4 = 6$ c) Where are you going?
2. Let p be "The sun is hot" and q be "The moon is made of cheese." Find the truth value of $p \land q$.
3. What is the only condition under which $p \to q$ is False?
4. What is the negation of: r: "I am 15 years old"?
5. Construct a truth table for $p \lor \sim p$. What kind of statement is this?

ASSIGNMENT (TAKE-HOME)

1. Let p = "He is tall" and q = "He is handsome." Write symbolically: a) He is tall and handsome. b) He is tall or he is not handsome. c) It is not true that he is short and handsome. d) He is handsome if and only if he is tall.
2. Construct a full truth table for $(\sim p \lor q) \to p$.
3. Construct a truth table for $(p \land q) \land \sim(p \lor q)$. What type of statement is this?
4. Given p = True, q = False, r = True, determine the truth value of $(p \land \sim q) \to (r \lor q)$.
5. Explain the difference between a contradiction and a contingency, giving one example each.