

FIRST TERM E –LEARNING NOTE

SUBJECT: FURTHER MATHEMATICS

CLASS: SS2

FIRST TERM SCHEME OF WORK

WEEK	TOPIC
1	Finding quadratic equation with given sum and product of roots, conditions for equal roots, real roots and no root
2	Tangents and Normals to Curves
3	Polynomials ;definition, basic operations $+$, $\times$, $-$, $\div$;--
4	Polynomials (Continued) factorization
5	Cubic Equation , roots of cubic equations
6	Review and Test
7	Logical Reasoning ; fundamental issues and definitions and theorem proving
8	Trigonometric Function , six trig functions of angles of any magnitude (sine, cosine,tangent,secant, cosecant, cotangent)
9	Relationship between graph of trigonometric ratios such as $\sin x$ and $\sin 2x$, graphs of $y = a \sin (bx) + c$, $y = a \cos (bx) + c$, $y = a \tan (bx) + c$
10	Graphs of inverse by ratio and equation of simpletrgonometric identities
11	Revision

REFERENCES

- Further Mathematics Project 1 by TuttuhAdegun
- Further Mathematics Project 2 by TuttuhAdegun
- Additional Mathematics by Godman

WEEK 1

TOPIC: SOLUTION TO QUADRATIC EQUATION

FINDING QUADRATIC EQUATION GIVEN SUM AND PRODUCT OF ROOTS CONDITION FOR EQUAL ROOTS, REAL ROOTS AND NO ROOT

We recall that if $ax^2 + bx + c = 0$, where a , b and c are constants such that $a \neq 0$, then,

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{Or } x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Suppose we represent these distinct roots by α and β ; thus:

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and

$$\beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

We may also put $D = b^2 - 4ac$, so that

$$\alpha = \frac{-b + \sqrt{D}}{2a}$$

$$\beta = \frac{-b - \sqrt{D}}{2a}$$

Sum of roots

$$\begin{aligned} \alpha + \beta &= \frac{(-b + \sqrt{D})}{2a} + \frac{(-b - \sqrt{D})}{2a} \\ &= \frac{-2b}{2a} \end{aligned}$$

$$= \frac{-b}{a}$$

Products of roots

$$\alpha\beta = \frac{(-b + \sqrt{D})(-b - \sqrt{D})}{2a \times 2a}$$

$$\therefore \alpha\beta = \frac{b^2 - D}{4a^2}$$

$$= \frac{b^2 - (b^2 - 4ac)}{4a^2}$$

$$= \frac{4ac}{4a^2}$$

$$= \frac{c}{a}$$

Hence, if $ax^2 + bx + c = 0$, where a , b and c are constants and $a \neq 0$ then $\alpha + \beta = \frac{-b}{a}$,

$\alpha\beta = \frac{c}{a}$, we recall from 5.3 that by the method of factorization if

$$x^2 + x - 42 = 0$$

$$\text{then } (x - 6)(x - 7) = 0$$

Hence the roots of the equation are 6 and -7. In general, if a quadratic equation factorizes into

$$(x - \alpha)(x - \beta) = 0$$

then α and β must be the roots of that equation.

The general quadratic equation $ax^2 + bx + c = 0$ can also be written as:

$$x^2 + \frac{bx}{a} + \frac{c}{a} = 0 \quad \dots(1)$$

If the roots of the equation are α and β then the above equation can be written as:

$$(x - \alpha)(x - \beta) = 0$$

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0 \quad \dots(2)$$

By comparing coefficients in equations (1) and (2)

$$-(\alpha + \beta) = \frac{b}{a}$$

$$\therefore \alpha + \beta = \frac{-b}{a}$$

$$\text{and } \alpha\beta = \frac{c}{a}$$

The above consideration gives rise to two problems:

- (a) Given a quadratic equation, we can find the sum and product of the roots.
- (b) Given the roots, we can formulate the corresponding quadratic equation.

The quadratic equation whose roots are α and β is

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

Find the sum and product of the roots of each of the following quadratic equations:

(a) $2x^2 + 3x - 1 = 0$

(b) $3x^2 - 5x - 2 = 0$

(c) $x^2 - 4x - 3 = 0$

(d) $\frac{1}{2}x^2 - 3x - 1 = 0$

Solution

(a) $2x^2 + 3x - 1 = 0$

$a = 2; b = 3; c = -1$

Let α and β be the roots of the equation, then

$$\alpha + \beta = \frac{-b}{a} = \frac{-3}{2}$$

$$\alpha\beta = \frac{c}{a} = \frac{-1}{2}$$

(b) $3x^2 - 5x - 2 = 0$

$a = 3; b = -5; c = -2$

Let α and β be the root of the equation, then

$$\alpha + \beta = \frac{-b}{a} = \frac{5}{3}$$

$$\alpha\beta = \frac{c}{a} = \frac{-2}{3}$$

(c) $x^2 - 4x - 3 = 0$

$a = 1; b = 4; c = -3$

Let α and β be the root of the equation, then

$$\alpha + \beta = \frac{-b}{a} = \frac{4}{1}$$

$$\alpha\beta = \frac{c}{a} = -3$$

(d) $\frac{1}{2}x^2 - 3x - 1 = 0$

$a = \frac{1}{2}, b = -3, c = -1$

Let α and β be the root of the equation, then

$$\alpha + \beta = \frac{-b}{a} = \frac{(3)}{\frac{1}{2}} = 6$$

$$\alpha\beta = \frac{c}{a} = \frac{-1}{\frac{1}{2}} = -2$$

Find the quadratic equation whose roots are:

(a) 3 and -2

(b) $\frac{1}{2}$ and 5

(c) -1 and 8

(d) $\frac{3}{4}$ and $\frac{1}{2}$

Solution

The quadratic equation whose roots are α and β is $x^2 - (\alpha + \beta)x + \alpha\beta = 0$.

(a) $\alpha + \beta = 3 - 2 = 1, \alpha\beta = 3(-2) = -6$

$\therefore$ The quadratic equation whose roots are 3 and -2 is $x^2 - x - 6 = 0$.

(b) $\alpha + \beta = \frac{1}{2} + 5 = \frac{11}{2}, \alpha\beta = \frac{5}{2}$

$\therefore$ The quadratic equation whose roots are $\frac{1}{2}$ and 5 is

$$x^2 - \frac{11}{2}x + \frac{5}{2} = 0$$

or $2x^2 - 11x + 5 = 0$

(c) $\alpha + \beta = 7, \alpha\beta = -8$

: $\alpha\beta = 7, \alpha\beta = -8$

: The quadratic equation whose roots are -1 and 8 is $x^2 - 7x - 8 = 0$.

(b) $\alpha + \beta = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}, \alpha\beta = \frac{3}{4} \times \frac{1}{2} = \frac{3}{8}$

: The quadratic equation whose roots are $\frac{3}{4}$ and $\frac{1}{2}$ is

$$x^2 - \frac{5}{4}x + \frac{3}{8} = 0$$

or $8x^2 - 10x + 3 = 0$

Symmetric Properties of Roots

of $ax^2 + bx + c = 0$, then

$$\alpha + \beta = -\frac{b}{a}; \alpha\beta = \frac{c}{a}$$

Certain relations involving α and β can also be determined from $\alpha + \beta$ and $\alpha\beta$ even when we do not know α and β distinctively. Such relations are usually said to be symmetric.

They are symmetric in the sense that if α and β are interchanged, either the relation remains the same or is multiplied by -1.

If $\alpha \neq \beta$, determine whether or not each of the following is symmetric:

(a) $\alpha + \beta$

(b) $\alpha\beta$

(c) $\alpha^2\beta^2$

(d) $\alpha^2 - \beta^2$

(e) $3\alpha + 2\beta$

(f) $\alpha^2\beta^2$

Solution

(a) $\alpha + \beta = \beta + \alpha$

: $\alpha + \beta$ is symmetric

(b) $\alpha\beta = \beta\alpha$

: $\alpha\beta$ is symmetric

(c) $\alpha^2\beta^2 = \alpha^2\beta^2$

: $\alpha^2\beta^2$ is symmetric

(d) $\alpha^2 - \beta^2 = -(\beta^2 - \alpha^2)$

: $\alpha^2 - \beta^2$ is symmetric

(e) $3\alpha + 2\beta \neq 3\beta + 2\alpha$ since $\alpha \neq \beta$

: $3\alpha + 2\beta$ is not symmetric

(f) $\alpha^2 + \beta^2 = \beta^2 + \alpha^2$

: $\alpha^2 + \beta^2$ is symmetric

If α and β are the roots of $3x^2 - 4x - 1 = 0$, find the value of:

(a) $\alpha + \beta$

(b) $\alpha\beta$

(c) $\alpha^2\beta^2$

(d) $\frac{1}{\alpha} + \frac{1}{\beta}$

(e) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

(f) $\alpha^3\beta^3$

(g) $\alpha - \beta$

(h) $\frac{1}{\alpha+1} + \frac{1}{\beta+1}$

Solution

$a = 3; b = -4; c = -1$

(a) $\alpha + \beta = -\frac{b}{a} = \frac{4}{3}$

(b) $\alpha\beta = \frac{c}{a} = -\frac{1}{3}$

(c) $\alpha^2\beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$
 $= \frac{16}{9} + \frac{2}{3} = \frac{22}{9}$

(d) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\beta + \alpha}{\alpha\beta} = \frac{4}{-\frac{1}{3}} = -4$

$$-\frac{1}{3}$$

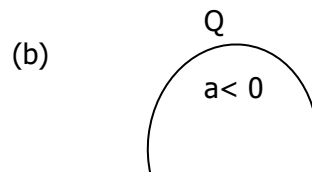
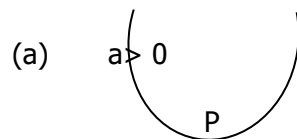
$$(e) \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{22}{9} = -\frac{22}{3}$$

$$(f) \alpha^3 \beta^3 = (\alpha + \beta)(\alpha^2 + \beta^2 - \alpha\beta) \\ = (\alpha + \beta)(\alpha^2 + \beta^2 - 3\alpha\beta) \\ = \frac{3}{4} \left[\frac{16}{9} + 1 \right] \\ = \frac{100}{27}$$

$$(g) \text{ We know that } (\alpha - \beta)^2 = \alpha^2 + \beta^2 - 2\alpha\beta \\ = (\alpha\beta)^2 - 4\alpha\beta \\ (\alpha - \beta) = \sqrt{(\alpha\beta)^2 - 4\alpha\beta} \\ = \frac{16}{9} \sqrt{\frac{4}{3}} \\ = \frac{28}{9} \sqrt{7}$$

$$(h) \frac{1}{\alpha+1} + \frac{1}{\beta+1} = \frac{\beta+1+\alpha+1}{(\alpha+1)(\beta+1)} \\ = \frac{\alpha+\beta+2}{\alpha\beta+\alpha+\beta+1} \\ = \frac{\frac{4}{3} + 2}{\frac{1}{3} + \frac{4}{3} + 1} \\ = \frac{\frac{10}{3}}{\frac{8}{3}} \\ = \frac{5}{4}$$

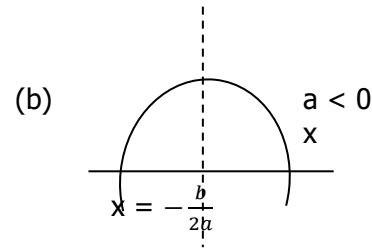
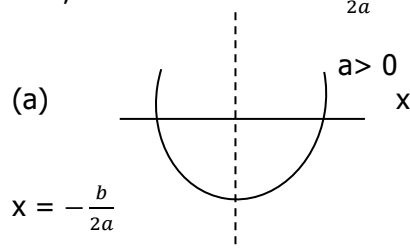
The Graph of $y = ax^2 + bx + c$ ($a \neq 0$) is called a parabola and has two shapes depending on whether $a > 0$ or $a < 0$.



When $a > 0$, the lowest point on the graph is called the minimum point, and it occurs when $x = -\frac{b}{2a}$. Similarly, when $a < 0$

the highest point Q on the graph is called the maximum point and it occurs when $x = -\frac{b}{2a}$

Also, the line when $x = -\frac{b}{2a}$ is a line of symmetry.



Nature of Roots

We recall that the solution of

$$ax^2 + bx + c = 0$$

$$\text{is } x = \frac{-b \pm \sqrt{D}}{2a}, \text{ where } D = b^2 - 4ac$$

Three restrictions can be placed on the value of D .

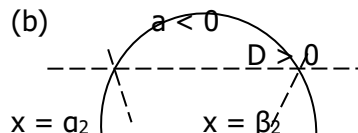
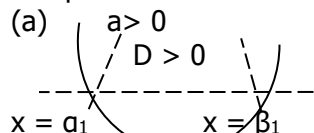
(a) $D > 0$

(b) $D < 0$

(c) $D = 0$

When $D > 0$

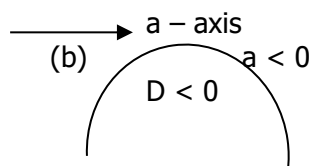
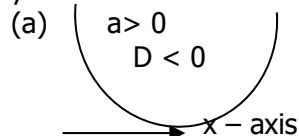
The roots of the equation are real and distinct. The graph of $y = ax^2 + bx + c$ crosses the x -axis at two points.



If in addition D is a perfect square, the roots are rational, but if D is not a perfect square, the roots are irrational and are always in conjugate pairs.

When $D < 0$

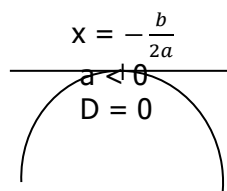
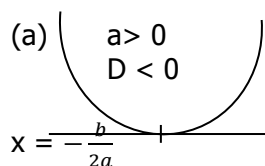
The roots are not real. They are said to be imaginary as $\sqrt{D}$ is not a real number. The graph of $y = ax^2 + bx + c$ does not cross the x -axis in this case.



When $D = 0$

The roots are real and equal. They are said to be coincidental. The graph touches the x -axis at

$$x = -\frac{b}{2a}$$



Since D enables us to determine the position of the graph of $y = ax^2 + bx + c$ relative to the x -axis, it is called a discriminant.

Determine the nature of roots of the following quadratic equations:

(i) $x^2 - 3x - 2 = 0$

(ii) $x^2 - 6x + 9 = 0$

(iii) $2x^2 - 2x + 5 = 0$

Solution

(i) $a = 1$; $b = -3$; $c = -2$

$$D = b^2 - 4ac$$

$$= 9 + 8$$

$$= 17 > 0$$

Hence the roots of the equation are real and distinct.

(ii) $x^2 - 2x + 9 = 0$

$a = 1$; $b = -2$; $c = 9$

$$D = b^2 - 4ac$$

$$= 4 - 36$$

$$= 0$$

Hence the roots are real and equal.

$$(iii) 2x^2 - 2x + 5 = 0$$

$$a = 2; b = -2; c = 5$$

$$D = b^2 - 4ac$$

$$= 4 - 40$$

$$= -36$$

Hence the roots are imaginary.

Evaluation

1. Find the quadratic equation where roots are

(a) 3 and -2

(b) $\frac{3}{4}$ and $\frac{1}{2}$

General Evaluation

(1) If α and β are the roots of $3x^2 - 4x - 1 = 0$, find the value of:

(a) $\alpha^2 + \beta^2$

(b) $\frac{1}{\alpha} + \frac{1}{\beta}$

(c) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

(d) $\alpha^3 + \beta^3$

(e) $\alpha - \beta$

(2) Find the sum and product of roots of these equation

(a) $2x^2 + 3x - 1 = 0$

(b) $3x^2 - 5x - 2 = 0$

Reading assignment

New Further Maths Project 2 page 8, 9, 10, 11

Weekend Assignment

(1) Determine the nature of roots of $x^2 - 3x - 2 = 0$

(a) Real

(b) Imaginary

(c) Equal

(d) Coincidental

(2) If $\alpha \neq \beta$ which of the following is not symmetric

(a) $\alpha\beta = \beta\alpha$

(b) $\alpha + \beta = \beta + \alpha$

(c) $3\alpha + 2\beta = 3\beta + 2\alpha$

(d) $\alpha^2 + \beta^2 = \beta^2 + \alpha^2$

If α and β are the roots of $2x^2 - 7x - 3 = 0$, find:

(3) $\alpha\beta^2 + \alpha^2\beta$

(a) $\frac{7}{2}$ (b) $\frac{3}{2}$ (c) $-\frac{21}{4}$ (d) $\frac{4}{21}$

(4) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

(a) $-\frac{61}{6}$ (b) $\frac{6}{61}$ (c) $\frac{31}{3}$ (d) $\frac{1}{6}$

(5) $\frac{1}{\alpha} + \frac{1}{\beta}$

(a) $-\frac{3}{7}$ (b) $-\frac{7}{3}$ (c) $\frac{3}{2}$ (d) $\frac{4}{7}$

Theory

(1) Find the constants p , q and r such that $3x^2 - 12x + 16 = p(x + q)^2 + r$

(2) If α and β are the roots of $x^2 - 10x + 2 = 0$, find $\alpha^3 - \beta^3$.

WEEK 2

TOPIC: Tangents and Normal to Curves

For any curve, $\frac{dy}{dx}$ is the gradient function. At any point on the curve, $\frac{dy}{dx}$ at that point, gives the gradient of the tangent at the point. The derivation of y with respect to x at $x = x_1$ is denoted.

$$\left. \frac{dy}{dx} \right|_{x=x_1}$$

Recall that the equation of the line of gradient m through (x_1, y_1) is $y - y_1 = m(x - x_1)$.

From this equation, we can easily obtain the equation of the tangent.

The straight line, perpendicular to the tangent at the point of contact of the tangent to the curve is called the **Normal** to the curve.

If m^1 is the gradient of the Normal, and m is the gradient of the tangent at the point of contact of the tangent to the curve, then

$$m^1 = \frac{-1}{m}$$

So at the point (x_1, y_1) the equation of the normal is $y - y_1 = \frac{-1}{m} (x - x_1)$

Find the equation of the tangent and the normal to the curve $y = 2x^3 - x^2 + 3x + 1$ at the point $x = 1$.

Solution

Given that $y = 2x^3 - x^2 + 3x + 1$

$$\frac{dy}{dx} = 6x^2 - 2x + 3$$

$$\left. \frac{dy}{dx} \right|_{x=1} = 6 - 2 + 3 = 7$$

If m is the gradient of the tangent at $x = 1$, then $m = 7$.

At the point $x = 1$; $y = 2 - 1 + 3 + 1 = 5$.

The equation of the tangent at the point $x = 1$ is

$$y - 5 = 7(x - 1)$$

$$y - 5 = 7x - 7$$

If m^1 is the gradient of the normal at $x = 1$, then

$$m^1 = \frac{-1}{m} = \frac{-1}{7}$$

Hence, the equation of the normal at $x = 1$, is $y - 5 = \frac{-1}{7} (x - 1)$

$$7(y - 5) = -1(x - 1)$$

$$7y - 35 = -x + 1$$

$$7y + x - 36 = 0$$

Find the equation of the tangent to the curve $x^2y + y^2x + 3x - 13 = 0$ at the point $(1, 2)$

Solution

$$x^2y + y^2x + 3x - 13 = 0$$

$$x^2 \frac{dy}{dx} + 2xy + 3y^2 x \frac{dy}{dx} + y^2 + 3 = 0$$

$$(x^2 + 3y^2x) \frac{dy}{dx} + 2xy + y^2 + 3 = 0$$

$$(x^2 + 3y^2x) \frac{dy}{dx} = -2xy - y^2 - 3$$

$$\frac{dy}{dx} = \frac{-2xy - y^2 - 3}{x^2 + 3y^2x}$$

$$\frac{dy}{dx} = \frac{-2xy - y^2 - 3}{x^2 + 3y^2x}$$

If m is the gradient of the tangent at the point $(1, 2)$ then

$$m = \left. \frac{dy}{dx} \right|_{x=1, y=2}$$

$$= \frac{-4 - 8 - 3}{1 + 12} = \frac{-15}{13}$$

The equation of the tangent is therefore

$$y - 2 = \frac{-15}{13} (x - 1)$$

$$13(y - 2) = -15(x - 1)$$

$$13y - 26 = -15x + 15$$

$$13y + 15x - 41 = 0$$

A curve is defined by

$f(x) = x^3 - 6x^2 - 15x - 1$. Find

- (i) the derivation $f(x)$ with respect to x ;
- (ii) the gradient of the curve at the point where $x = 1$;
- (iii) the maximum and the minimum points.

Find the maximum and minimum points of the curve $y = x^3 - x^2 - 5x$ and sketch the curve.

A curve passes through the point (1, 0) and its gradient at any point $p(x, y)$ is $3x^2 - 1$.

- (i) Find the equation of the curve;
- (ii) Sketch the curve, indicating all turning points and the point of intersection with the axes.

Sketch the curve $y = x^3 - 6x^2 + 9x$.

Find:

- (i) the equation of the tangent to the curve $y = x^3 + x^2 - 8x + 2$ at the point $A(1, -4)$;
- (ii) the coordinate of the point where the tangent meet the x - axis.

Evaluation

Find the equation of the normal to each of the following curve

- (4) $y = (2x - 3)(x + 2)$ at $x = 1$

General Evaluation

Find the equation of the tangent to each of the following curve at the given points

- (1) $y = x^2 - 3x - 4$ at $x = 1$
- (2) $y = x^3 + 2x^2 - 3x + 1$ at $x = -1$
- (3) $y = 1 - 2x + 5x^2 - x^3$ at $x = 3$

Find the equation of the normal to each of the following curve

- (5) $y = 6 - 2x + 3x^2 - 2x^3$ at $x = 0$

Reading Assignment

New Further Maths Project 2 page 143, 144, 169

Weekend Assignment

A curve $y = 4x^3 - 2x^2 + 7x + 5$ at point $x = 3$, find the

- (1) Gradient of its tangent

- (a) 12 (b) 108 (c) 103
- (d) 115

- (2) Gradient of its normal

- (a) $\frac{-1}{12}$ (b) $\frac{-1}{108}$ (c) $\frac{-1}{103}$ (d) $\frac{-1}{115}$

- (3) Equation of the tangent

- (a) $y = 108x - 116$ (b) $y = 103x - 193$ (c) $y = 115x - 90$
- (d) $y = 12x - 309$

- (4) Equation of the normal

- (a) $103y + x - 11 = 0$ (b) $103x + y - 116 = 0$ (c) $116x + x - 119 = 0$
- (d) $116x + y - 103 = 0$

- (5) Find the gradient of this curve $y = 2x^2 - 5x + 8$ at point $x = 1$

- (a) 1 (b) -1 (c) 2 (d) 3

Theory

- (1) Find the gradient of the tangent and the normal of the curve $y = x^3 - 6x^2 - 15x - 1$ at point $x = 1$

- (2) A curve passes through the point (1, 0) and its gradient at any point $p(x, y)$ is $3x^2 - 1$, find the equation of the curve.

WEEK 3

TOPIC: Polynomials

Consider the expressions formed from the sum of integral non negative powers of a variable x taken together with some numerical constants. Such expressions are called **Polynomials**.

The general polynomial takes the form $a_n X^n + a_{n-1} X^{n-1} + \dots + a_2 X^2 + a_1 X + a_0$ where $a_n, a_{n-1}, \dots, a_0$ ($a_n \neq 0$) are numerical constants.

The numerical constants $a_n, a_{n-1}, \dots, a_2, a_1$ are called coefficients of $X^n, X^{n-1}, \dots, X^2, X$ respectively while a_0 is called the constant term of the polynomial.

The highest power of the variable is n and is called the degree of the polynomial. Let us designate the general polynomial by $p(x)$. Thus:

$$P(x) = a_n X^n + a_{n-1} X^{n-1} + \dots + a_2 X^2 + a_1 X + a_0.$$

The following are examples of polynomials:

(a) $P_1(x) = 3x^2 - 2x + 4$

(b) $P_2(x) = 3x^4 - 2x^2 + x - 1$

(c) $P_3(x) = x + 1$

(d) $P_4(x) = 2x^3 + x - 3$

The following are not polynomials:

(a) $f_1(x) = (x^2 + 2x - 3)$

(b) $f_1(x) = 3x^2 - 4x^2 + 2x - 1$

$$2x + 3$$

(c) $f_1(x) = (2x - 3)^1$

Equality of Polynomials

The polynomial $p(x) = a_n X^n + a_{n-1} X^{n-1} + \dots + a_2 X^2 + a_1 X + a_0$ is said to be equal to the polynomial.

$$Q(x) = b_n X^n + b_{n-1} X^{n-1} + \dots + b_2 X^2 + b_1 X + b_0 \text{ provided}$$

$$a_n = b_n, a_{n-1} = b_{n-1}, \dots, a_2 = b_2, a_1 = b_1, a_0 = b_0$$

Addition and Subtraction of Polynomials

Let $P(x) = a_n X^n + a_{n-1} X^{n-1} + \dots + a_2 X^2 + a_1 X + a_0$

$Q(x) = b_n X^n + b_{n-1} X^{n-1} + \dots + b_2 X^2 + b_1 X + b_0$ then,

$$P(x) + Q(x) = (a_n + b_n) X^n + (a_{n-1} + b_{n-1}) X^{n-1} + \dots + (a_2 + b_2) X^2 + (a_1 + b_1) X + a_0 + b_0.$$

Also,

$$P(x) - Q(x) = (a_n - b_n) X^n - (a_{n-1} - b_{n-1}) X^{n-1} - \dots + (a_2 - b_2) X^2 + (a_1 - b_1) X + a_0 - b_0.$$

Given that $P_1(x) = 7x^3 - 4x^2 + 3x + 4$;

$P_2(x) = 5x^2 + 6x + 1$ and $P_3(x) = 4x^3 + 2x - 3$.

Find:

(a) $P_1(x) + P_2(x)$

(b) $P_1(x) + P_3(x)$

(c) $P_1(x) - P_2(x)$

(d) $P_3(x) - P_2(x)$

(e) $P_1(x) + P_2(x) + P_3(x)$

Solution

(a) $P_1(x) + P_2(x)$

$$= (7x^3 - 4x^2 + 3x + 4) + (5x^2 + 6x + 1)$$

$$= 7x^3 + (-4x^2 + 5x^2) + (3x + 6x) + (4 + 1)$$

$$= 7x^3 + x^2 + 9x + 5$$

(b) $P_1(x) + P_3(x)$

$$(7x^3 - 4x^2 + 3x + 4) + (4x^3 + 2x - 3)$$

$$= (7x^3 + 4x^3) + (-4x^2) + (3x + 2x) + (4 - 3)$$

$$= 11x^3 - 4x^2 + 5x + 1$$

(c) $P_1(x) - P_2(x)$

$$= (7x^3 - 4x^2 + 3x + 4) - (5x^2 + 6x + 1)$$

$$= 7x^3 + (-4x^2 - 5x^2) + (3x - 6x) + (4 - 1)$$

$$= 7x^3 - 9x^2 - 3x + 3$$

(d) $P_3(x) - P_2(x)$

$$\begin{aligned}
&= (4x^3 + 2x - 3) - (5x^2 + 6x + 1) \\
&= (4x^3 - 5x^2 + (2x - 6x) + (-3 - 1)) \\
&= 4x^3 - 5x^2 - 4x - 4 \\
&(e) P_1(x) + P_2(x) + P_3(x) \\
&= (7x^3 - 4x^2 + 3x + 4) + (5x^2 + 6x + 1) + (4x^3 + 2x - 3) \\
&= (7x^3 + 4x^3) + (-4x^2 + 5x^2) + (3x + 6x + 2x) + (4 + 1 - 3) \\
&= 11x^3 + x^2 + 11x + 2
\end{aligned}$$

Given that $P_1(x) = 2x^3 + 4x^2 - x + 1$

$P_2(x) = 3x^4 + x^3 - 2x^2 + x - 3$ and

$P_3(x) = 4x^3 + 2x - 4$

Find:

- (a) $2P_1(x) + P_2(x)$
- (b) $3P_2(x) + 2P_3(x)$
- (c) $3P_1(x) - 3P_3(x)$
- (d) $P_3(x) + 2P_1(x) - 3P_2(x)$

Solution

$$\begin{aligned}
&(a) 2P_1(x) + P_2(x) \\
&= 2(2x^3 + 4x^2 - x + 1) + (3x^4 + x^3 - 2x^2 + x - 3) \\
&= (4x^3 + 8x^2 - 2x + 2) + (3x^4 + x^3 - 2x^2 + x - 3) \\
&= 3x^4 + 5x^3 + 6x^2 - x - 1 \\
&(b) 3P_2(x) + 2P_3(x) \\
&= 3(3x^4 + x^3 - 2x^2 + x - 3) + 2(4x^3 + 2x - 4) \\
&= 9x^4 + 3x^3 - 6x^2 + 3x - 9 + 8x^3 + 4x - 8 \\
&= 9x^4 + 11x^3 - 6x^2 + 7x - 17 \\
&(c) 3P_1(x) - 3P_3(x) \\
&= 3(2x^3 + 4x^2 - x + 1) - 3(4x^3 + 2x - 4) \\
&= 6x^3 + 12x^2 - 3x + 3 - 12x^3 - 6x + 12 \\
&= -6x^3 + 12x^2 - 9x + 15 \\
&(d) P_3(x) + 2P_1(x) - 3P_2(x) \\
&= (4x^3 + 2x - 4) + 2(2x^3 + 4x^2 - x + 1) - 3(3x^4 + x^3 - 2x^2 + x - 3) \\
&= 4x^3 + 2x - 4 + 4x^3 + 8x^2 - 2x + 2 - 9x^4 - 3x^3 + 6x^2 - 3x + 9 \\
&: P_3(x) + 2P_1(x) - 3P_2(x) = -9x^4 + 5x^3 + 14x^2 - 3x + 7
\end{aligned}$$

The value of $p(x)$ at $x = a$ is denoted by $p(a)$ and is obtained by substituting a for x in the polynomial.

Example

Given that $p(x) = 2x^3 + 5x^2 - 9x - 18$, find;

- (a) $P(1)$
- (b) $P(-1)$
- (c) $P(2)$
- (d) $P(0)$

Solution

$$\begin{aligned}
&(a) P(1) = 2(1)^3 + 5(1)^2 - 9(1) - 18 \\
&\quad = 2 + 5 - 9 - 18 \\
&\quad = -20 \\
&(b) P(-1) = 2(-1)^3 + 5(-1)^2 - 9(-1) - 18 \\
&\quad = -2 + 5 + 9 - 18 \\
&\quad = -6 \\
&(c) P(2) = 2(2)^3 + 5(2)^2 - 9(2) - 18 \\
&\quad = 16 + 20 - 18 - 18 \\
&\quad = 0 \\
&(d) P(0) = 2(0)^3 + 5(0)^2 - 9(0) - 18 \\
&\quad = -18
\end{aligned}$$

Multiplication of polynomials

The product of two polynomials of degrees m and n is another polynomial of degree $m + n$.

Example

Given that $P_1(x) = 2x^2 + 5x + 6$ and $P_2(x) = 3x^2 - 2x + 1$, find P_1P_2

Solution

Method 1

$P_1 \times P_2$

$$\begin{aligned} &= (2x^2 + 5x + 6)(3x^2 - 2x + 1) \\ &= 2x^2(3x^2 - 2x + 1) + 5x(3x^2 - 2x + 1) + 6(3x^2 - 2x + 1) \\ &= 6x^4 - 4x^3 + 2x^2 + 15x^3 - 10x^2 + 5x + 18x^2 - 12x + 6 \\ &= 6x^4 + (4x^3 + 15x^3) + (2x^2 - 10x^2 + 18x^2) + 5x - 12x + 6 \\ &= 6x^4 + 11x^3 + 10x^2 - 7x + 6 \end{aligned}$$

Method 2

$$\begin{array}{r} 2x^2 + 5x + 6 \\ 3x^2 - 2x + 1 \\ \hline 2x^2 + 5x + 6 \\ -4x^3 - 10x^2 - 12x \\ \hline 6x^4 + 15x^3 + 18x^2 \\ \hline 6x^4 + 11x^3 + 10x^2 - 7x + 6 \end{array}$$

Method 2 is usually called Long Multiplication method.

Given that $P_1(x) = 4x^3 - 2x^2 + 3x - 1$ and $P_2(x) = 3x^2 - 4$ find $P_1(x) \times P_2(x)$

Solution

Method 1

$$\begin{aligned} P_1P_2 &= (3x^2 - 4)(4x^3 - 2x^2 - 3x - 1) \\ &= 3x^2(4x^3 - 2x^2 + 3x - 1) \\ &\quad - 4(4x^3 - 2x^2 + 3x - 1) \\ &= 12x^5 - 6x^4 + 9x^3 - 3x^2 - 16x^3 + 8x^2 - 12x + 4 \\ &= 12x^5 - 6x^4 - 7x^3 + 5x^2 - 12x + 4 \end{aligned}$$

Method 2

$$\begin{array}{r} 4x^3 - 2x^2 - 3x - 1 \\ 3x^2 - 4 \\ \hline -16x^3 + 8x^2 - 12x + 4 \\ \hline 12x^5 - 6x^4 - 9x^3 + 3x^2 \\ \hline 12x^5 - 6x^4 - 7x^3 + 5x^2 - 12x + 4 \end{array}$$

Division of Polynomials

A polynomial of degree n can be divided by another polynomial of degree m if $n \geq m$.

Divide the polynomial $P(x) = 3x^2 - 2x + 4$ by the polynomial $P(x) = x + 2$

Solution

Since P_1 is being divided by P_2 it is called the dividend while P_2 is called the divisor. The result of division of P_1 by P_2 is called the quotient and whatever is left after division is called the remainder. The procedure of division of P_1 by P_2 can best be demonstrated step by step as follows:

Step 1

Divide the first term of the dividend by the first term of the divisor to obtain the first term of the quotient.

$$x + 2 \overline{) 3x^2 - 2x + 4} \quad \begin{array}{l} 3x \\ \hline \end{array}$$

Step 2

Multiply each of the terms of the divisor by the quotient.

$$x + 2 \overline{) \begin{array}{r} 3x \\ 3x^2 - 2x + 4 \\ 3x^2 + 6x \end{array}}$$

Step 3

Subtract the product obtained in step 2 from the first two terms of the dividend and add the next term of the dividend.

$$x + 2 \overline{) \begin{array}{r} 3x \\ 3x^2 - 2x + 4 \\ 3x^2 + 6x \\ -8x + 4 \end{array}}$$

Step 4

Using $-8x + 4$ as a new dividend repeat step 1, 2 and 3.

$$\begin{array}{r} 3x - 8(c) \\ x + 2 \overline{) \begin{array}{r} 3x^2 - 2x + 4(b) \\ 3x^2 + 6x \\ -8x + 4 \\ -8x - 16 \\ \hline 20(d) \end{array}} \end{array}$$

Note that:

(a) $x + 2$ is the divisor;

(b) $3x^2 - 2x + 4$ is the dividend;

(c) $3x - 8$ is the quotient

(d) 20 is the remainder

The expression $3x^2 - 2x + 4$ can be written as:

$$3x^2 - 2x + 4 = (x + 2)(3x - 8) + 20$$

$\downarrow$ $\downarrow$ $\downarrow$ $\downarrow$
 Dividend Divisor Quotient Remainder

In general $P(x) = D(x) \times Q(x) + R$

$P(x)$ = Dividend

$D(x)$ = Divisor

$Q(x)$ = Quotient

R = Remainder

Divide $4x^3 + 6x^2 - 2x + 7$ by $2x - 3$ and hence find the quotient and the remainder

Solution

$$2x - 3 \overline{) \begin{array}{r} 2x^2 + 6x + 8 \\ 4x^3 + 6x^2 - 2x + 7 \\ 4x^3 + 6x^2 \\ \hline 12x^2 - 2x \\ 12x^2 - 18x \\ \hline 16x + 7 \\ 16x - 24 \\ \hline 31 \end{array}}$$

Quotient = $2x^2 + 6x + 8$

Remainder = 31

Find the quotient and remainder when $2x^4 - 3x^3 + x^2 - 4x + 5$ is divided by $x^2 + 3x + 1$

Solution

$$x^2 + 3x + 1 \overline{) \begin{array}{r} 2x^2 - 9x + 26 \\ 2x^4 - 3x^3 + x^2 - 4x + 5 \\ 2x^4 - 6x^3 + 2x^2 \\ \hline -9x^3 - x^2 - 4x \\ -9x^3 - 27x^2 - 9x \\ \hline 26x^2 + 5x + 5 \\ 26x^2 + 78x + 26 \\ \hline \end{array}}$$

Hence, Quotient = $2x^2 - 9x + 26$
 Remainder = $-73x - 21$

Find the quotient and remainder when $x^3 + 8$ is divided by $x^2 - 2x + 4$

Solution

$$\begin{array}{r}
 x^2 - 2x + 4 \overline{) x^3 + 8} \\
 \underline{x^3 - 2x^2 + 4x} \\
 2x^2 + 4x + 8 \\
 \underline{2x^2 + 4x + 8} \\
 0
 \end{array}$$

Quotient = $x + 2$

Remainder = 0

Hence, $x^3 - 2x^2 + 4x$ is a factor of $x^3 + 8$.

The Remainder Theorem

The Long Division Method. Although a little cumbersome enables us to find, not only the quotient, but the remainder as well. Suppose we are given the polynomial $f(x) = 2x^3 - 3x^2 + 4x - 1$ and we wish to find the remainders when $f(x)$ is divided by:

(a) $x - 1$

(b) $x - 2$

(c) $x - 3$

By using the long division method we have:

$$\begin{array}{r}
 x - 1 \overline{) 2x^3 - 3x^2 + 4x - 1} \\
 \underline{2x^2 - 2x^2} \\
 -x^2 + 4x \\
 \underline{-x^2 + x} \\
 3x - 1 \\
 \underline{3x - 3} \\
 2
 \end{array}$$

So when $f(x)$ is divided by $x - 1$, the remainder is 2.

$$\begin{array}{r}
 x - 2 \overline{) 2x^3 - 3x^2 + 4x - 1} \\
 \underline{2x^3 - 4x^2} \\
 -x^2 + 4x \\
 \underline{-x^2 + 2x} \\
 6x - 1 \\
 \underline{6x - 12} \\
 11
 \end{array}$$

So when $f(x)$ is divided by $x - 2$, the remainder is 11.

$$\begin{array}{r}
 x - 3 \overline{) 2x^3 - 3x^2 + 4x - 1} \\
 \underline{2x^3 - 6x^2} \\
 3x^2 + 4x \\
 \underline{-x^2 + 9x} \\
 13x - 1 \\
 \underline{13x - 39} \\
 38
 \end{array}$$

So when $f(x)$ is divided by $x - 3$, the remainder is 38.

Now, find:

- a. $f(1)$
- b. $f(2)$
- c. $f(3)$
- d. $f(1) = 2 - 3 + 4 - 1 = 2$
- e. $f(2) = 16 - 12 + 8 - 1 = 11$
- f. $f(3) = 54 - 27 + 12 - 1 = 38$

Compare the answers in each of the following pairs

- i. (a) and (d)
- ii. (b) and (e)
- iii. (c) and (f)

What do you notice?

This is a remarkable result and it is not a mere coincidence.

Let us try another one.

Given that $H(x) = x^3 + 2x^3 - 4x + 3$, find the remainder when $H(x)$ is divided by

- (a) $x + 1$
- (b) $x + 2$
- (c) $x + 3$

$$\begin{array}{r} \text{(a)} \quad x + 1 \quad \begin{array}{r} x^2 + x - 5 \\ \overline{x^3 + 2x^3 - 4x + 3} \\ x^3 + x^2 \\ \hline x^2 - 4x \\ x^2 + x \\ \hline -5x + 3 \\ -5x - 5 \\ \hline 8 \end{array} \end{array}$$

When $H(x)$ is divided by $x + 1$, the remainder is 8.

$$\begin{array}{r} \text{(b)} \quad x + 2 \quad \begin{array}{r} x^2 \quad -4 \\ \overline{x^3 + 2x^3 - 4x + 3} \\ x^3 + 2x^2 \\ \hline -4x + 3 \\ -4 - 8 \\ \hline 11 \end{array} \end{array}$$

So, when $H(x)$ is divided by $x + 2$, the remainder is 11.

$$\begin{array}{r} \text{(c)} \quad x + 3 \quad \begin{array}{r} x^2 + x - 1 \\ \overline{x^3 + 2x^3 - 4x + 3} \\ x^3 + 3x^2 \\ \hline x^2 - 4x \\ -x^2 + 3x \\ \hline -x + 3 \\ -x - 3 \\ \hline 6 \end{array} \end{array}$$

When $H(x)$ is divided by $x + 3$, the remainder is 6. Now, find:

- (d) $H(-1)$
- (e) $H(-2)$
- (f) $H(-3)$
- (d) $H(-1) = -1 + 2 + 4 + 3 = 8$
- (e) $H(-2) = -8 + 8 + 8 + 3 = 11$
- (f) $H(-3) = -27 + 18 + 12 + 3 = 6$

Compare again the answers in each of the following pairs:

- (i) (a) and (d)

(ii) (b) and (e)

(iii) (c) and (f)

What again do you notice?

(a) When $H(x)$ is divided by $x + 1$ the remainder is $H(-1)$

(b) When $H(x)$ is divided by $x + 2$ the remainder is $H(-2)$

(c) When $H(x)$ is divided by $x + 3$ the remainder is $H(-3)$

So we can safely conclude that if $H(x)$ is divided by $x - a$ the remainder is $H(a)$ and this forms the basis of what is called the remainder theorem.

Theorem

If the polynomial $f(x)$ is divided by $x - a$, the remainder is $f(a)$.

Proof

The polynomial function $f(x)$ can be written as:

$$f(x) = (x - a) Q(x) + R \quad \dots(1)$$

where $x - a$ is the divisor, $Q(x)$ the quotient and R the remainder.

Put $x = a$ in (1)

$$f(a) = (a - a) Q(a) + R$$

$$f(a) = R$$

The theorem whose proof we have just established, is called the **Remainder Theorem**. In general, if $f(x)$ is divided by $ax + b$ then the remainder is

$$f\left(\frac{-b}{a}\right)$$

A special case of the remainder theorem is when $f(x)$ leaves no remainder when it is divided by $x - a$. We therefore say that $x - a$ is a factor of $f(x)$. The modified theorem is called, **Factor Theorem** and it states:

If $f(a) = 0$ then $x - a$ is a factor of $f(x)$.

Example

Given that $f(x) = 3x^3 - 4x^2 + 2x + 3$, find the remainder when $f(x)$ is divided by $x - 1$.

Solution

Let R be the remainder. Then using the remainder theorem, $R = f(1)$

$$\begin{aligned} f(1) &= 3(1)^3 - 4(1)^2 + 2(1) + 3 \\ &= 3 - 4 + 2 + 3 \\ &= 4 \end{aligned}$$

Find the remainder when

$f(x) = (x + 3)(x - 2)(x + 2)$ is divided by $x + 1$.

Solution

Let R be the remainder when $f(x)$ is divided by $(x + 1)$ then $R = f(-1)$

$$\begin{aligned} f(-1) &= (-1 + 3)(-1 - 2)(-1 + 2) \\ &= (2)(-3)(+1) \\ &= -6 \end{aligned}$$

Hence the remainder is -6

Find the remainder when $f(x) = 2x^3 + 3x^2 - 4x + 1$ is divided by $2x - 1$. What conclusion can you draw?

Solution

Let R be the remainder when $f(x)$ is divided by $2x - 1$ then $R = f\left(\frac{1}{2}\right)$

$$\begin{aligned} f\left(\frac{1}{2}\right) &= 2 f\left(\frac{1}{2}\right)^3 + 3 f\left(\frac{1}{2}\right)^2 - 4 f\left(\frac{1}{2}\right) + 1 \\ &= \frac{1}{4} + \frac{3}{4} - 2 + 1 \\ &= 0 \end{aligned}$$

Hence $2x - 1$ is a factor of $f(x)$.

Evaluation

1. If $g(x)$ is the quotient when $f(x) = 2x^3 + 3x^2 - 2x + 1$ is divided by $x+2$ find the remainder when $g(x)$ is divided by $x-1$

General Evaluation

If $f(x) = 6x^3 + 13x^2 + 2x - 5$,

- (a) show that $f(-1) = 0$ (b) find the factor of $f(x)$
 (1) When the polynomial $f(x) = x^4 + px^3 + x^2 + qx + 1$ is divided by $x^2 + 3x + 2$ quotient is $x^2 - 1$ and the remainder is $5x + 3$. Find the value of constants p and q .
 (2) If $(x + 1)$ is a factor of the polynomial $f(x) = x^3 + kx^2 + 3x + 10$. Find the value of the constant k , and factorise the polynomial completely.

Reading assignment

New Further Maths Project 1 pages 71 – 75 Exercise 6b Q1, 8, 22 and 26

Weekend Assignment

1. Given that $f(x) = x^5 + 4x^4 - 6x^2 + 2x + 2$, find $f(-1)$
 (a) 2 (b) -3 (c) -4 (d) -2
 (2) Determine the values of p and q if $(x - 1)$ and $(x - 2)$ are factors of $2x^3 + px - 4 + q$
 (a) $p = 12, q = 13$ (b) $p = -13, q = -12$ (c) $p = -13, q = 12$
 (d) $p = 10, q = 8$
 (3) Find zero of the polynomial $p(x) = x^3 + 4x^2 + x - 6$
 (a) $x = -1, 2, 3$ (b) $x = -1, -3$ (c) $x = 1, -2, -3$ (d) $x = 1, 2, -3$
 (4) If $f(x) = 6x^3 + 13x^2 + 2x - 5$, find the factors of $f(x)$
 (a) $(x - 1)(2x - 1)(3x - 5)$ (b) $(x + 1)(2x - 1)(3x + 5)$ (c) $(x + 1)(2x + 1)(3x - 5)$
 (d) $(x + 1)(1 - 2x)(5 - 3x)$
 (5) If $(2x + 1)$ is a factor of the polynomial $f(x) = 2x^3 - 8x + x^2 + k$, find the value of the constant k . (a) -2 (b) -3 (c) -4 (d) 3

Theory

- (1) Find the values of the constant p, q and r such that, when the polynomial $f(x) = x^3 + px^2 + qx + r$ is divided by $(x + 2), (x - 1)$ and $(x - 3)$, the remainder are respectively -48, 0 and 2. Hence factorise $f(x)$ completely.
 (2) If $x - 2$ is a factor of $f(x) = x^3 - x^2 + px + q$ and $f(x)$ leaves a remainder of 12 when it is divided by $(x - 3)$, find
 (a) the values of the constant p and q
 (b) the three values of c for which $x^3 - x^2 + px + q = 0$

$$\frac{8x - 8}{0}$$

$$\begin{aligned}\text{Now, } x^2 - 6x + 8 &= x^2 - 4x - 2x + 8 \\ &= (x - 2)(x - 4)\end{aligned}$$

$$\text{Hence, } f(x) = (x - 1)(x - 2)(x - 4)$$

What must be subtracted from $f(x) = x^2 + 2x^2 - 3x + 5$ so that it will be exactly divisible by $x - 2$?
Hence, find that polynomial $H(x)$ which is exactly divisible by $x - 2$ and factorise it completely.

Solution

$$f(x) = x^2 + 2x^2 - 3x + 5$$

$$\begin{aligned}f(2) &= 8 + 8 - 6 + 5 \\ &= 15\end{aligned}$$

:When $f(x)$ is divided by $x - 2$ the remainder is 15. Hence 15 must be subtracted from $f(x)$ to make it exactly divisible by $x - 2$.

Let the new polynomial which is exactly divisible by $x - 2$ be $H(x)$, then:

$$\begin{aligned}H(x) &= f(x) - 15 \\ &= x^2 + 2x^2 - 3x + 5 - 15 \\ &= x^2 + 2x^2 - 3x - 10\end{aligned}$$

Using long division

$$\begin{array}{r}x^2 + 4x + 5 \\ x - 2 \overline{) x^2 + 2x^2 - 3x - 10} \\ \underline{x^2 + 2x^2} \\ 4x^2 - 3x \\ \underline{4x^2 - 8x} \\ 5x - 10 \\ \underline{5x - 10} \\ 0\end{array}$$

But $x^2 + 4x + 5$ is not factorizable, hence $H(x) = (x - 2)(x^2 + 4x + 5)$.

Given that $f(x - 1) = x^2 + 3x - 1$, find $f(3)$.

$$\text{Put } x + 1 = 3$$

$$x = 2$$

$$\begin{aligned}: f(3) &= (2)^2 + 3(2) - 1 \\ &= 4 + 6 - 1 \\ &= 9\end{aligned}$$

(1) Given that $p_1(x) = 5x^3 + 3x^2 - 2x + 6$, $p_2(x) = x^3 + 4x^2 - 3x + 1$ and $p_3(x) = 2x^3 - 3x + 2$, find

(i) $(p_2 - p_3) p_1$ (ii) $p_2(p_1 + p_3)$

(2) Divide (i) $4x^3 + 3x^2 - 2x + 1$ by $x^2 + 2x - 1$
obtain the
remainder and quotient.

(ii) $x^4 + 2x + 3$ by $x^2 - 1$ to

Evaluation

Find the quadratic equation whose roots are (i) 3 & -2 (ii) -1 & 8 (iii) $\frac{3}{4}$ & $\frac{1}{2}$

General Evaluation

(1) When $x^2 + bx + 2$ is divided by $x + 3$ the remainder is 5. Find the value of b .

(2) If $2x^2 - (b - 4)x - 4(b + 2) = 0$, has equal roots, find the possible values of b .

(3) Factorise completely $x^3 + 5x^2 - 3x + 1$

(4) Solve the following pair of equation simultaneously $4x - 3y = 17$, $3x^2 - 2y^2 + x - 4y = 73$

Reading Assignment

Further Maths 1 pages 66 – 69 Exercise 6a Q5, 9 and 10

Weekend Assignment

- (1) Find the remainder when $2x^3 - 4x^2 + x - 3$ is divided by $x + 3$
(a) 84 (b) -86 (c) -64 (d) 76
- (2) Factorise $x^4 - 1$ completely (a) $(x + 2)(x - 3)$ (b) $(x + 1)(x - 1)(x^2 + 1)$
(c) $(x + 1)(x + 2)(x - 3)$ (d) $(x^2 + 1)(x + 1)(x - 2)$
- (3) Given that $p_1(x) = 2x^4 + 3x^3 - x^2 + 2x - 3$ and $p_1(x) = 3x^3 + 2x + 2$. Find $3p_1(x) - 3p_1(x)$
(a) $6x^4 - 3x^2 - 15$ (b) $5x^4 - 3x^4 + 2x^2 - x + 3$ (c) $6x^4 = 9x^3 - 15$
(d) $6x^4 + 18x^2 + 6x^2$
- (4) Given that $p(x) = x^3 + 4x^2 - 3x + 1$, find $p(\frac{1}{2})$
(a) $\frac{3}{4}$ (b) $\frac{6}{7}$ (c) $\frac{5}{8}$ (d) 2.
- (5) Given that $p(x) = ax^2 + bx + 1$, $p(\frac{1}{2}) = \frac{1}{2}$ and $p(-2) = 23$, determine the values of a and b
(a) $a = -3, b = 4$ (b) $a = 2, b = -3$ (c) $a = 4, b = -3$ (d) $a = 3, b = -3$

Theory

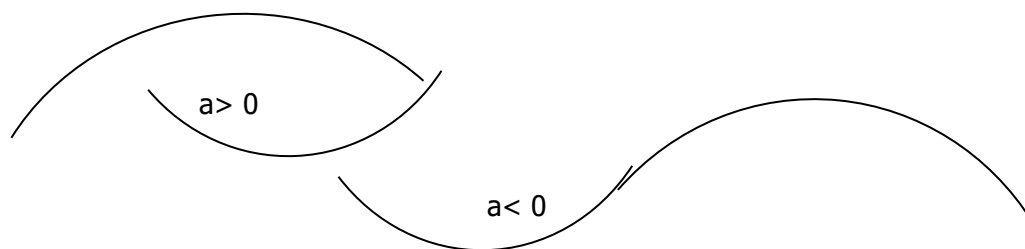
- (1) Find the quotient and remainder when $2x^4 - 3x^3 + x^2 - 4x + 5$ is divided by $x^2 + 3x + 1$
- (2) If $p_1 = 3x^3 + 2x^2 - x + 2$, $p_2 = 2x^2 + x - 6$ and $p_3 = x^3 + 3x^2 + 2x - 4$, find
(i) $p_3(p_1 + p_2)$ (ii) $p_2 + p_3 - 3p_1$ (iii) $p_2 \times (p_3 + p_1)$

WEEK 5

TOPIC: Cubic equations and their factorization , graphs of cubic equations

Polynomials of degree three have the general form $y = ax^3 + bx^2 + cx + d$ ($a \neq 0$). The curve is usually called a **cubical parabola**.

A cubical parabola has two shapes depending on whether $a > 0$ or $a < 0$.



Sketch each of the following curves represented by the following functions:

(a) $y = x^3 + 2x^2 - 5x - 6$

(b) $y = 12 + 4x - 3x^2 - x^3$

Solution

(a) $y = x^3 + 2x^2 - 5x - 6$

Using the factor theorem and long division the expression can be factorized as:

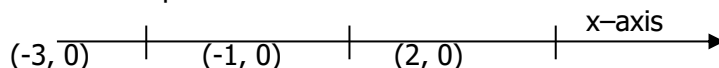
$$y = (x - 2)(x + 1)(x + 3)$$

The zeros of the polynomial are therefore $x = 2$, $x = -1$ and $x = -3$, hence the x – intercepts are $(2, 0)$, $(-1, 0)$ and $(-3, 0)$.

The y – intercept is $(0, -6)$.

Next, we shall consider the behaviour of the function at different intervals along the x – axis. This will enable us to see whether the curve is above or below the axis.

Mark the x – intercepts on the x – axis.



The intervals we shall consider are:

(a) $x < -3$

(b) $-3 < x < -1$

(c) $-1 < x < 2$

(d) $x > 2$

We shall examine the signs (+ve or –ve) in each of the intervals.

$x = -4$ is in the intervals $x < -3$

$$f(-4) = -18 < 0$$

Hence the part of the graph in the interval $x < -3$ is below the axis.

$x = -2$ is in the interval $-3 < x < -1$

$$f(-2) = 4 > 0$$

Hence the part of the graph in the interval $-3 < x < 2$ is above the x – axis $x = 0$ is in the interval $-1 < x < 2$

$$f(0) = -6 < 0.$$

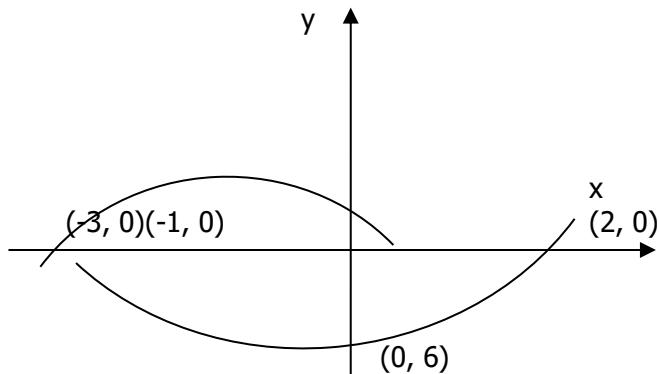
Hence the part of the graph in the interval $-1 < x < 2$ is below the x – axis.

$x = 3$ is in the interval $x > 2$

$$f(3) = 24 > 0$$

Hence the part of the graph in the interval is above the x – axis.

The intercept on the axis coupled with the behaviour of the function at different intervals on the x – axis will enable us to get the shape of the curve.



(b) $y = 12 + 4x - 3x^2 - x^3$

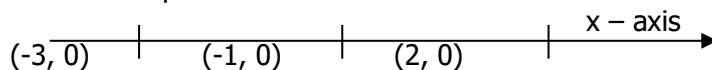
Using the factor theorem and long division method

$$y = (2 - x)(2 + x)(3 + x)$$

The zeros of the polynomial are $x = 2$, $x = -2$ and $x = -3$, hence the x – intercepts are $(2, 0)$, $(-2, 0)$ and $(-3, 0)$.

The y – intercept is $(0, 12)$.

Mark the x – intercept on the axis.



The intervals we shall consider are:

(a) $x < -3$

(b) $-3 < x < -1$

(c) $-1 < x < 2$

(d) $x > 2$

$x = -4$ is in the interval $x < -3$

$$f(-4) = 12 < 0,$$

hence the part of the graph within this interval is above the x – axis.

$x = -2.5$ is in the interval $-3 < x < -2$

$$f(-2.5) = \frac{9}{8} < 0,$$

hence the part of the graph within this interval $-3 < x < -2$ is above the axis.

$x = 0$ is in the interval $-2 < x < 2$

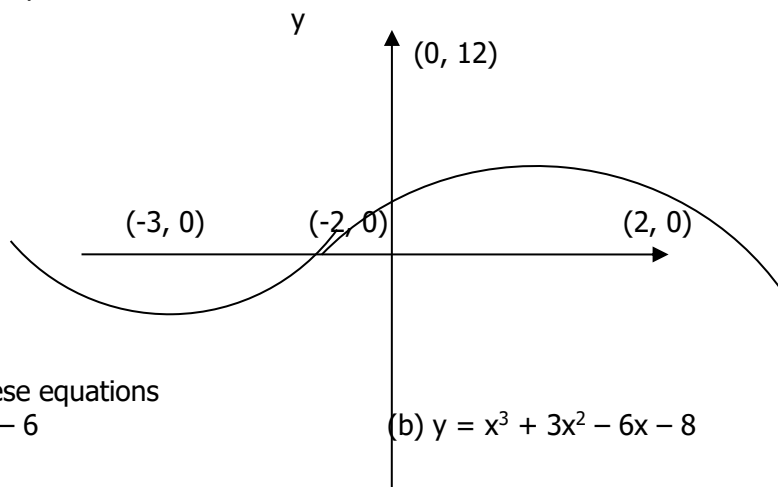
$$f(0) = 12 > 0,$$

hence the part of the graph within this interval $-2 < x < 2$ is above the axis.

$x = 3$ is in the interval $x > 2$

$$f(3) = -30 < 0,$$

hence the part of the graph within the interval $x > 2$ is below the x – axis.



Evaluation

Sketch the curve of these equations

(a) $y = x^3 - 6x^2 + 11x - 6$

(b) $y = x^3 + 3x^2 - 6x - 8$

General Evaluation

(1) Factorise the following completely

(a) $x^3 + 10x^2 + 23x + 14$

(b) $x^4 - 1$

Weekend Assignment

Given that a cubic equation $x^3 + 2x^2 - 19x - 20 = 0$ has 4 as one its roots, find the

- (1) Second root (a) -1 (b) 1 (c) 2 (d) 3
(2) Third root (a) 5 (b) -8 (c) 3 (d) 2
(3) Sum of the second and third roots (a) -4 (b) 6 (c) 4 (d) -6
(4) Product of the second and third roots (a) 6 (b) 5 (c) -6 (d) -5
(5) Find the zeros of $x^2 - 1$ (a) 2 or -2 (b) 1 or 2 (c) -1 or 1 (d) 1 or -2

Theory

- (1) If $(x + 1)$ is a factor of $f(x) = x^3 + kx^2 + 3x + 10$, find the value of the constant k .
(2) Factorise $f(x)$ completely.

WEEK 6

Review

WEEK 1 FINDING QUADRATIC EQUATIONS WITH GIVEN SUM AND PRODUCT OF ROOTS

WEEK 2 TANGENT AND NORMAL TO CURVES

WEEK 3 POLYNOMIAL

WEEK 4 POLYNOMIAL (CONTINUED)

WEEK 5 CUBIC EQUATIONS

WEEK 7

TOPIC: Logical reasoning

Statements

A statement in a logical context is a declaration, verbal or written that is either true or false but not both.

A true statement is said to have a truth value **T**, while a false statement is said to have a truth value **F**.

Example 1

The following are statements:

- (a) Nigeria is an African country.
- (b) The earth is conical in shape.
- (c) If I run I shall not late.
- (d) Japanese are hard working people.

Example 2

The following are not statements in the logical context.

- (a) Who is he?
- (b) What a lovely man!
- (c) Take the pencil away.
- (d) If I think of my family.

In general, questions, exclamations, commands and expressions of feelings which cannot be assigned a truth value T or F are not statements in the logical context.

By convention, we shall use letter $P, q, r, \dots$ to denote statements.

Negation

Given a statement P , the negation of P , written $\neg P$ is the statement; "it is false that P " or "nor P ".

If P is true, $\neg P$ is false and if P is false $\neg P$ is true. In other words, if P has the truth value T then $\neg P$ has the truth value F and if P has the truth value F then $\neg P$ has the truth value T.

The relationship between P and $\neg P$ can be summarized in the following table.

Table 1

P	$\neg P$
T	F
F	T

Table 1 is called a truth table.

If P is a statement: "Nigeria is a rich country" then $\neg P$ is the statement: "it is false that Nigeria is a rich country" or in a more reasonable English "Nigeria is not a rich country".

Let q be the statement "some lawyers are honest people" then $\neg q$ is the statement "it is false that some lawyers are honest people". In a more reasonable English, we can also write $\neg q$ as some lawyers are not honest people.

Let P be the statement $3 + 4 = 8$ then $\neg P$ is the statement $3 + 4 \neq 8$.

Let q be the statement $X + 1 \geq 4$ then $\neg q$ is the statement $X + 1 < 4$. /

Let P be the statement "The set of numbers 2, 4, 6, 8, ... is a set of even numbers" then $\neg P$ is the statement "The set of numbers 2, 4, 6, 8, ... is a set of odd numbers."

1. State which of the following are statements in the logical context:

- (a) Caesar was a great leader.
- (b) Stop talking to the boys.
- (c) Decide whether you are going to the club's meeting now.
- (d) Oh Mansa Musa, you are wonderful!
- (e) The Broking House in Ibadan, is a magnificent building.

2. State which of the following are statements in the logical context:

- (a) As old as Methuselah.
- (b) The set of numbers 3, 5 and 4 is not a Pythagorean triplet.
- (c) Is he a serious teacher at all?
- (d) If 6 is an odd number, then $3 + 5 = 10$.

3. Write the negation of each of the following statements:

- (a) He is a handsome man.
- (b) It is very cold in Siberia.
- (c) It is very hot in tropics.
- (d) The sky is blue.

4. Write the negation of each of the following statements:

- (a) The party leader will win the election.
- (b) The football captain scored the first goal.
- (c) Short cuts are dangerous.
- (d) Honest men are very rare to come by.

5. Statement with reasons whether the statement q is a negation of the statement P in each of the following:

- (a) p . The line AB is parallel to the line CD,
 q . The line AB is perpendicular to the line CD.
- (b) p . He is a good leader.
 q . He is a bad leader.
- (c) p . She is a good leader.
 q . She is a good follower.

(6) Write the negation of each of the following avoiding the word 'not' as much as possible.

- (a) The car is moving fast.
- (b) He was present in school yesterday.
- (c) The Equator is a Great circle.
- (d) His friend is younger than my brother.

(7) Write the negation of each of the following avoiding the word 'not' as much as possible.

- (a) He obtained the least mark in the examination.
- (b) She is the shortest girl in the class.
- (c) He is an ugly man.
- (d) The hospital is in a bad state.

Conditional Statement

Let p stand for the statement 'Lagos is a state in Nigeria' and q stand for the statement 'Lagos is a state in Africa'. One way the two statements can be combined is 'If Lagos is a city in Nigeria then Lagos is a city in Africa' or 'if p and q '.

The statement "if p and q is a combination of two simple statements p and q it is therefore called a **compound statement**.

Symbolically, we can write the compound statement if p then q as $p \Rightarrow q$.

The statement $p \Rightarrow q$ is read as:

p implies q or
 If p then q or
 q is necessary for p or
 p is sufficient for q or
 p only if q or
 p follows from q or
 q if p .

The symbol $\Rightarrow$ is an operation. In the compound statement $p \Rightarrow q$, p is called the **antecedent** while the sub statement q is called the **consequent** of $p \Rightarrow q$.

The truth or falsity of the implication $p \Rightarrow q$ is illustrated in Table 2.

Table 2

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

The statement $p \Rightarrow q$ is false if the antecedent is true and the consequent is false. The statement $p \Rightarrow q$ is sometimes called a **Conditional Statement**.

Consider the following statements,

- (a) If Cairo is in Africa then 8 is an even number.
- (b) If Cairo is in Africa then 8 is an odd number.
- (c) If Cairo is in Asia then 8 is an even number.
- (d) If Cairo is in Asia then 8 is an odd number.

The statements (a) (c) and (d) are all true but the statement (b) is not true for the simple reason that the antecedent is true while the consequent is false. Note that although in statement (d) both the antecedent and consequent are false, yet the whole statement is true.

Converse Statement

Let p be the statement 'a triangle is equilateral' and q the statement 'a triangle is equiangular'. The statement p implies q , i.e. $p \Rightarrow q$. The statement q also implies p , i.e. $q \Rightarrow p$. In other words, if a triangle is equiangular then it is equilateral. The statement $q \Rightarrow p$ is called the **converse of the statement**

$p \Rightarrow q$

Inverse Statement

If p is the statement 'a triangle is equilateral' and q is the statement 'a triangle is equiangular' the statement $\neg p$ is the statement 'if a triangle is not equilateral then it is not equiangular'.

The statement $\neg p \Rightarrow \neg q$ is called the **inverse of the statement** $p \Rightarrow q$.

Contrapositive Statement

If p is the statement 'a triangle is equilateral' and q is the statement 'a triangle is equiangular' the statement $\neg q$ is 'if a triangle is not equiangular then it is not equilateral'. The statement $\neg q \Rightarrow \neg p$ is called the **contrapositive statement** of $p \Rightarrow q$.

Biconditional Statement

Let p be the statement 'the interior angles of a polygon are equal' and q be the statement 'a polygon is regular'.

$p \Rightarrow q$ is the statement 'If the interior angles of a polygon are equal then the polygon is regular'

$q \Rightarrow p$ is the statement 'If a polygon is regular then the interior angles of the polygon are equal'.

We see here that $p \Rightarrow q$ and $q \Rightarrow p$.

The two conditional statements are valid. We say that p and q imply 'each other' or p is equivalent to q and we write $p \Leftrightarrow q$. The statement $p \Leftrightarrow q$ is called a **Biconditional Statement** of p and q and the

statements p and q are said to be equivalent to each other.

Other terminologies for $p \implies q$ are:

q is equivalent to p

p is necessary and sufficient for q

q if and only if p

p if and only if q

If p then q and if q then p .

The truth of falsity of $p \implies q$ is completely illustrated by Table 3.

Table 3

p	q	$p \implies q$
T	T	T
T	F	F
F	T	T
F	F	T

So a biconditional statement is true when the two substatements have the same truth value.

Consider the following four statements:

(a) Nyerere is an African name if and only if a, e, i, o, u are vowels;

(b) Nyerere is an African name if and only if a, e, i, o, u are consonants;

(c) Nyerere is a European name if and only if a, e, i, o, u are vowels;

(d) Nyerere is a European name if and only if a, e, i, o, u are are consonants.

The statements (a) and (d) are both true since the substatements of each have the same truth value. The statements (b) and (c) are false since the substatement of each have different truth values.

The Chain Rule

If p , q and r are three statements such that $p \implies q$ and $q \implies r$ then $p \implies r$. This is called the **chain rule** and it may have several links.

Consider the argument:

T₁: If a student works very hard, he passes his examination.

T₂: If a student passes his examination, he is awarded a certificate.

T₃: If a student works very hard, he is awarded a certificate.

Let P be the statement 'a student works very hard'.

Let q be the statement 'a student passes his examination'.

Let r be the statement 'a student is awarded a certificate'.

The argument has the following structural form:

$p \implies q$, $q \implies r$: $p \implies r$

This argument follows the chain rule link hence it is said to be valid.

The statement T₁ and T₂ are called the **Premises** while the statement T₃ is called the **Conclusion** of the argument.

The argument is valid not on the basis of the truth or falsity of its premises but on the basis of the structural form from which follows the chain rule link.

Compound Statement

A statement may consist of two or more simple statements or substatements. Such a statement is called

a **Compound** or **Composite Statement**. For example, the compound statement: 'John is an intelligent and courageous boy', consists of the substatements: 'John is an intelligent boy' and 'John is a courageous boy'.

By convention, we shall use letters p , q , r ,... to denote statements.

We proceed now to introduce logical symbols called **Connectives**.

Disjunction

Two statements can be combined by the use of the connective 'or'.

The statement: 'He is a philosopher' can be combined with the statement 'He is a teacher as follow: 'He is a philosopher or he is a teacher'. In a more refined English, the combined statement is 'Either he is a philosopher, or he is a teacher'.

If the statement 'He is a philosopher' is denoted p and the statement 'He is a teacher' is denoted q , the combined statement is either p or q or simply q or q . In symbolic logic, q or q is designated $p \vee q$, where the connective $p \vee q$. Where the connective $\vee$ means 'or'.

The word 'or'; in English Language is used in two different sense.

In the statement, 'the professor will deliver the lecture at Bayero University or remain at the hotel'. The 'or' is used in an exclusive sense, in that it is not possible for the professor to deliver lecture at Bayero University, and at the same time remain at the hotel. The 'or' is said to be used in the exclusive sense.

In the statement, 'He is a good politician or a good statement', the or is used in the inclusive sense, in that it is possible for the good politician to be a good statesman as well. We shall clear this apparent ambiguity, it we settle for one meaning.

In this text, we use 'or' in the inclusive sense. The statement $p \vee q$ can be read; either p or q or both or simply $p \vee q$ or both. The truth table for $p \vee q$ is illustrated in Table 4

Table 4

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The statement $p \vee q$ is false when both p and q are false, otherwise $p \vee q$ is true.

Consider the following statements:

- (a) Cairo is in Africa or 8 is an even number.
- (b) Cairo is in Africa or 8 is an odd number.
- (c) Cairo is in Asia or 8 is an even number.
- (d) Cairo is in Asia or 8 is an odd number.

The statements (a) (b) and (c) are all true since at least one of the substatements is true. The statement (d) is false since the two substatements are false.

Conjunction

Another way by which two statements can be combined with the statement 'she is proud' using the connective 'and' as 'she is pretty and she is proud'. In a more refined English, the combined statement is: 'she is pretty and proud'.

If the statement 'she is pretty' is denoted p and the statement 'she is proud' is denoted q , the statement 'she is pretty and proud' can be written as p and q . In symbolic logic, p and q is designated $p \wedge q$, where

the connective $\wedge$ means 'and'.

The truth table for $p \wedge q$ is illustrated in Table 5

Table 5

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

The statement $p \wedge q$ is true when the sub statements p and q are both true, otherwise $p \wedge q$ is false.

Consider the following statements:

- (a) Cairo is in Africa and 8 is an even number.

(b) Cairo is in Africa and 8 is an odd number.

(c) Cairo is in Asia and 8 is an even number.

(d) Cairo is in Asia and 8 is an odd number.

The statements (a) (b) and (c) are all true since at least one of the substatements is true. The statement (d) is false since the two substatements are false

Of the four given statements, only (a) is true. The statements (b), (c) and (d) are false since at least one of their substatement is false.

Equivalent Statements

Two compound statements are said to be logically equivalent if they have the same truth value.

Use the truth table to establish that:

$$\neg(p \vee q) = \neg p \wedge \neg q$$

Solution

Table 6

p	q	$p \wedge q$	$\neg(p \vee q)$
T	T	F	F
T	F	F	F
F	T	F	F
F	F	T	T

(a)

p	q	p	$\neg q$	$\neg p \wedge \neg q$
T	T	F	F	F
T	F	F	F	F
F	T	T	F	F
F	F	T	T	T

(b)

We observe that the last columns of Table 6(a) and Table 6(b) have the same truth values.

Hence $\neg(p \vee q) = \neg p \wedge \neg q$

Use the truth table to prove that:

$$\neg(p \wedge q) = \neg p \vee \neg q$$

Solution

Table 7

p	q	$p \wedge q$	$\neg(p \wedge q)$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

(a)

p	q	p	$\neg q$	$\neg p \vee \neg q$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

(b)

The last columns of the two tables have the same truth values, hence $\neg(p \wedge q) = \neg p \vee \neg q$

Use the truth table to show that:

(a) the connective $\wedge$ distributes over the connective $\vee$;

(b) the connective $\vee$ distributes over the connective $\wedge$.

Solution

Given the statements p , q and r and the connectives $\wedge$ and $\vee$, we wish specifically to show that:

(a) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

(b) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

(a)

Table 8

p	q	r	$q \vee r$	$p \wedge (q \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	F	F	F	F
F	F	T	T	F
F	T	T	T	F
F	T	F	T	F

Table 9

p	q	r	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	T	T	T
T	F	F	F	F	F
F	F	F	F	F	F
F	F	T	T	F	F
F	T	F	T	F	F
F	T	T	T	F	F

We notice that the truth values of the two tables

Hence $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

(b)

Table 10

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	T
T	F	F	F	T
F	F	F	F	F
F	F	T	F	F
F	T	F	F	F
F	T	T	T	T

Table 11

p	q	r	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T
T	T	F	T	T	T
T	F	T	T	T	T
T	F	F	T	T	T
F	F	F	F	F	F
F	F	T	F	T	F
F	T	F	T	F	F
F	T	T	T	T	T

We see here also that the last two columns of Table 10 and Table 11 have the same truth value, hence

$$p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$$

Given the statements p , q and r show that:

(a) $p \wedge (q \wedge r) = (p \wedge q) \wedge r$

(b) $p \vee (q \vee r) = (p \vee q) \vee r$

Table 12

(a)

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$
T	T	T	T	T
T	T	F	F	F
T	F	T	F	F
T	F	F	F	F
F	F	F	F	F
F	F	T	F	F
F	T	F	F	F
F	T	T	T	F

(a)

(b)

P	Q	r	$p \wedge q$	$(p \wedge q) \wedge r$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	F
T	F	F	F	F
F	F	F	F	F
F	F	T	F	F
F	T	F	F	F
F	T	T	F	F

The last two columns of Table 12(a) and Table 12(b) are identical, hence;

$$p \wedge (q \wedge r) = (p \wedge q) \wedge r$$

We see here that the conjunctive connective as an operator, is associative.

(b)

Table 13

P	Q	r	$p \wedge q$	$(p \vee q) \vee r$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	T
T	F	F	F	T
F	F	F	F	F
F	F	T	F	T
F	T	F	F	T
F	T	T	T	T

p	q	r	$q \vee r$	$p \vee (q \vee r)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	T
F	F	F	F	F
F	F	T	T	T
F	T	F	T	T
F	T	T	T	T

Table last two columns of hence $p \vee (q \vee r)$

Table 13a and Table 13b are identical,

$$r) = (p \vee q) \vee r$$

The disjunctive connective as an operator is seen as being associative.

Tautology and Contradiction

A compound statement which is always true irrespective of the truth values of the substatements, is called a **Tautology**. A tautology is represented as **T**.

A compound statement which is always false, irrespective of the truth values of the substatements is called a **Contradiction**. A contradiction is usually represented as **F**.

Use the truth table to show that the statement $p \vee \neg p$ is a tautology.

Solution

Table 14

p	$\neg p$	$p \vee \neg p$
T	F	T
T	F	T
F	T	T
F	T	T

We observe that the last column of the table has the truth value T irrespective of the truth values of the substatements. Hence the statement $p \vee \neg p$ is a tautology.

Use the truth table to show the statement $p \wedge \neg p$ is a contradiction.

Solution

Table 15

p	$\neg p$	$p \wedge \neg p$
T	F	F
T	F	F
F	T	F
F	T	F

The last column of Table 15 has the truth value F irrespective of the truth values of the substatements hence, $p \wedge \neg p = F$.

Show that the statement

$p \wedge ((\neg p \wedge q) \vee (\neg p \wedge \neg q))$ is a contradiction.

Table 16

p	q	$\neg p$	$\neg q$	$\neg p \wedge q$	$\neg p \wedge \neg q$	$(\neg p \wedge q) \vee (\neg p \wedge \neg q)$	$p \wedge ((\neg p \wedge q) \vee (\neg p \wedge \neg q))$
T	T	F	F	F	F	F	F
T	F	F	T	F	F	F	F
F	T	T	F	T	F	T	F
F	F	T	T	F	T	T	F

We see that the truth value of expression in the last column of Table 16 is F irrespective of the truth values of the substatements, hence $p \wedge ((\neg p \wedge q) \vee (\neg p \wedge \neg q)) = F$

Laws of the Algebra of Logical Statements

There is a close relationship between the algebra of sets and algebra of logical statements. The logical connectives as operations obey the laws of algebra.

Commutative Laws

i. (a) $p \wedge q = q \wedge p$

I (b) $p \vee q = q \vee p$

Associative Laws

ii (a) $p \wedge (q \wedge r) = (p \wedge q) \wedge r$

ii (b) $p \vee (q \vee r) = (p \vee q) \vee r$

Distributive Laws

iii. (a) $p \wedge (q \vee r) = (p \wedge q) \vee (p \wedge r)$

iii. (b) $p \vee (q \wedge r) = (p \vee q) \wedge (p \vee r)$

Laws of Absorption

iv. (a) $p \wedge (p \vee q) = p$

iv. (b) $p \vee (p \wedge q) = p$

Idempotent Laws

v. (a) $p \wedge p = p$

v. (b) $p \vee p = p$

De Morgan's Laws

vi. (a) $\neg(p \wedge q) = \neg p \vee \neg q$

vi. (b) $\neg(p \vee q) = \neg p \wedge \neg q$

Laws of Complementation

vii. (a) $p \wedge \neg p = F$

(b) $p \vee \neg p = T$

vii. (a) $\neg F = T$

(b) $\neg T = F$

ix. $\neg(\neg p) = p$

Laws of Contraposition

x. $p \Rightarrow q = \neg q \Rightarrow \neg p$

Laws of Identity

xi. $P \wedge T = P$

xii. $P \wedge F = F$

xiii. $P \vee T = T$

xiv. $P \vee F = P$

All these laws can be verified using the truth table technique.

Using the truth table technique, show that if p , q and r are arbitrary statements, then:

$$(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r) = T$$

Solution**Table 17**

p	q	r	$p \Rightarrow q$	$q \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r)$	$p \Rightarrow r$	$(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	F	F	T	T
T	F	F	F	F	F	F	T
F	F	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	T	T	T	T	T	T	T

The truth value of the expression in the last column is T, hence

$$(p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r) = T$$

Use the truth table technique to show that

$$p \Rightarrow q = (p \Rightarrow q) \wedge (q \Rightarrow p) \Rightarrow$$

Solution**Table 18**

(a)

p	q	$p \Rightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

(b)

p	Q	$p \Rightarrow q$	$q \Rightarrow p$	$(p \Rightarrow q) \wedge (q \Rightarrow p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

The truth values of the expression in the last columns of each of the table are identical, hence

$$p \Rightarrow q = (p \Rightarrow q) \wedge (q \Rightarrow p)$$

Use the truth table technique to show that

$$p \Rightarrow \neg q = q \Rightarrow p$$

Solution

Table 19

		p	q	$p \Rightarrow q$	$\neg q$	$p \Rightarrow \neg q$
T	T	F	F	T	T	T
T	T	F	T	T	F	F
F	F	T	F	F	T	F
F	T	T	T	T	F	F

(b)

The last column of the two tables are identical, hence

$$p \Rightarrow \neg q = q \Rightarrow p$$

1. Let P be the statement 'He is funny' and q be the statement 'He is serious'. Write each of the following in a simple English:

(a) $p \vee q$

(b) $p \wedge q$

(c) $p \wedge \neg q$

(d) $\neg p \vee \neg q$

2. Let p be the statement 'she is beautiful' and q be the statement 'she is soft - spoken'. Write each of the following in symbolic form:

(a) She is beautiful and soft – spoken.

(b) Either she is beautiful or she is soft – spoken.

(c) She is beautiful but not soft – spoken.

(d) She is ugly but soft – spoken.

Evaluation

1. Find the truth value of these statements (a) If $11 > 8$ then $-1 < -8$ (b) If $3 + 4 \neq 10$ then $2 + 3 \neq 5$.

General Evaluation

(1) Let P be the statement: "He is funny" and q be the statement: "He is serious". Write each of the following in simple English (i) $p \vee q$ (b) $p \wedge \neg q$ (c) $\neg p \vee \neg q$

(2) If p and q represent two statements 'he is good in physics' and 'he is good in mathematics' respectively. Write the following in symbolic form; "he is good in physics if and only if he is good in mathematics".

Reading Assignment

F/Maths Project 1 pages 126 – 130 Exercise 9b Q 2, 3 and 4

Weekend Assignment

Objective

P is the statement "Ayo has determination and q is the statement "Ayo will succeed". Use this information to answer the questions. Which of these symbols represent these statements?

- (1) Ayo has no determination (a) $p \vee q$ (b) $p \Rightarrow q$ (c) $p \Rightarrow \neg q$
 (2) If Ayo has no determination then he won't succeed (a) $\neg p \vee \neg q$ (b) $p \vee \neg q$ (c) $\neg p \vee q$ (d) $p \vee \neg p$
 (3) If Ayo won't succeed then he has no determination (a) $\neg q \vee p$ (b) $\neg q \vee \neg p$ (c) $\neg q \vee p$ (d) $q \vee p$
 (4) If Ayo has determination then he will succeed (a) $p \vee q$ (b) $p \vee \neg q$ (c) $\neg p \vee \neg q$ (d) $p \vee q$
 (5) If Ayo has no determination then he will succeed (a) $\neg p \vee q$ (b) $\neg p \vee \neg q$ (c) $p \vee \neg q$ (d) $\neg p \vee \neg q$

Theory

- (1) Write down the inverse, converse and contrapositive of each of these statements.
 (i) If the bank workers work hard they will be adequately compensated.
 (ii) If he is humble and prayerful, he will meet with God's favour.
 (iii) If he set a good example, he will get a good followership.
 (2) Consider the following statements P: some dogs are tame Q: all tame animals are small.
 Which of the following is a valid conclusion from the above statements?
 (i) All dogs are tame (ii) No dog is small (iii) All small animals are tame (iv) Some dogs are small (v) All tame animals are dogs.

WEEK 8

TOPIC: Trigonometric functions

The basic trigonometric ratios can be defined in two ways:

- (i) traditional definition;
- (ii) modern definition.

Traditional Definition

The basic trigonometric ratios can be defined in terms of the sides of a right – angled triangle.

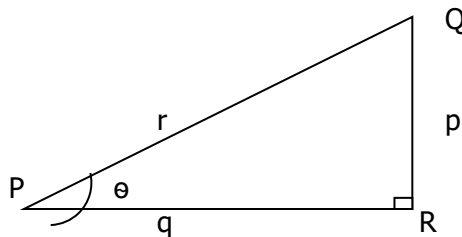


Fig. 14.3

ΔPQR in Fig 14.3 is a right angled triangle with $\angle QPR = \theta$ and $\angle PRQ = 90^\circ$. We define the three basic ratios as follows:

$$\text{Cosine of angle } \theta = \frac{PR}{PQ} = \frac{q}{r}$$

$$\text{Sine of angle } \theta = \frac{QR}{PQ} = \frac{p}{r}$$

$$\text{Tangent of } \theta = \frac{QR}{PR} = \frac{p}{q}$$

The cosine of angle θ , sine of angle θ and the tangent of angle θ will be abbreviated $\cos\theta$, $\sin\theta$ and $\tan\theta$ respectively.

Thus:

$$\cos\theta = \frac{q}{r}$$

$$\sin\theta = \frac{p}{r}$$

$$\tan\theta = \frac{p}{q}$$

Also:

$$\frac{\sin\theta}{\cos\theta} = \frac{p/r}{q/r} = \frac{p}{q} = \tan\theta$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

Ratios of the General Angle

First Quadrant

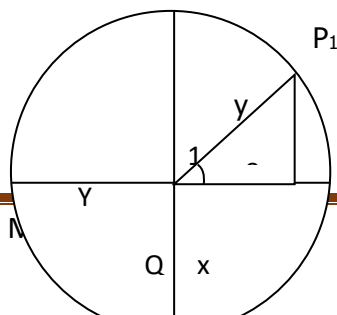


Fig 14.7

In Fig. 14.7 ΔOP_1N_1 is a right – angled triangle constructed from a unit circle.

$$OP_1 = 1$$

$$P_1N_1 = y$$

$$ON_1 = x$$

$$P_1ON_1 = \theta_1$$

$$\sin \theta_1 = y$$

$$\cos \theta_1 = x$$

$$\tan \theta_1 = \frac{y}{x}$$

Second Quadrant

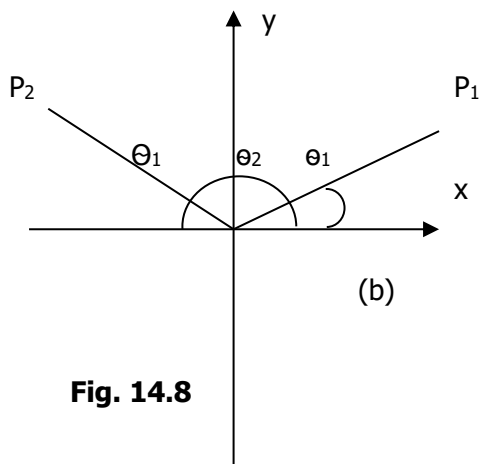
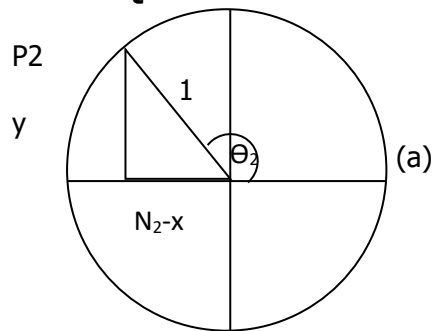


Fig. 14.8

From Fig. 14.8(a)

$$\sin \theta_2 = y$$

$$\cos \theta_2 = -x$$

$$\tan \theta_2 = \frac{y}{-x} = \frac{-y}{x}$$

From Fig. 14.8(b)

$$\theta_1 + \theta_2 = 180^\circ$$

$$\theta_2 = 180^\circ - \theta_1$$

$$\therefore \sin \theta_2 = \sin(180^\circ - \theta_1) = y = \sin \theta_1$$

$$\therefore \sin(180^\circ - \theta) = \sin \theta$$

$$\therefore \cos \theta_2 = \cos(180^\circ - \theta_1) = -x = -\cos \theta_1$$

$$\therefore \cos(180^\circ - \theta) = -\cos \theta$$

Similarly,

$$\tan \theta_2 = \tan (180^\circ - \theta_1) = \frac{y}{x} = -\tan \theta_1$$

$$: \tan (180^\circ - \theta) = -\tan \theta$$

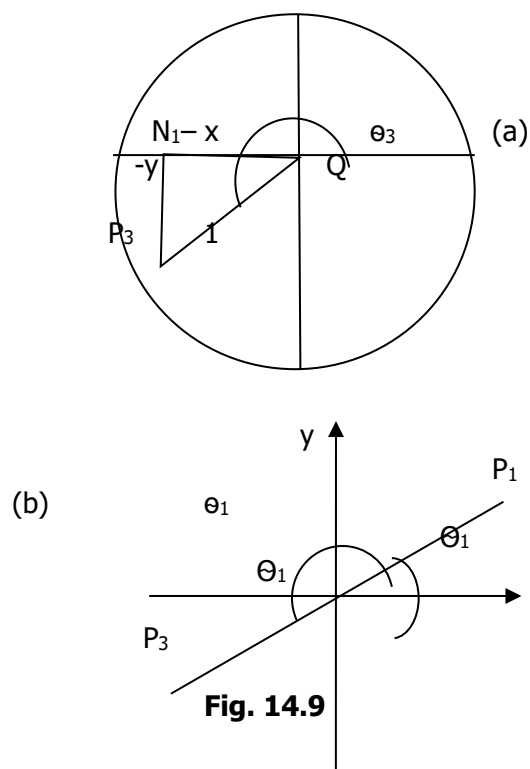
Hence in the second quadrant:

$$\sin(180^\circ - \theta) = \sin \theta$$

$$\cos(180^\circ - \theta) = -\cos \theta$$

$$\tan(180^\circ - \theta) = -\tan \theta$$

Third Quadrant



$$\sin \theta_3 = -y$$

$$\cos \theta_3 = -x$$

$$\tan \theta_3 = \frac{\sin \theta_3}{\cos \theta_3} = \frac{-y}{-x} = \frac{y}{x}$$

From Fig. 14.9(b)

$$\theta_3 = 180^\circ + \theta_1$$

$$\sin \theta_3 = \sin(180^\circ + \theta) = -y = -\sin \theta_1$$

$$: \sin(180^\circ + \theta) = -\sin \theta$$

$$\cos \theta_3 = \cos(180^\circ + \theta) = -x = -\cos \theta_1$$

$$: \cos(180^\circ + \theta) = -\cos \theta$$

Similarly,

$$\tan \theta_3 = \frac{\sin \theta_3}{\cos \theta_3} = \frac{-y}{-x} = \tan \theta_1$$

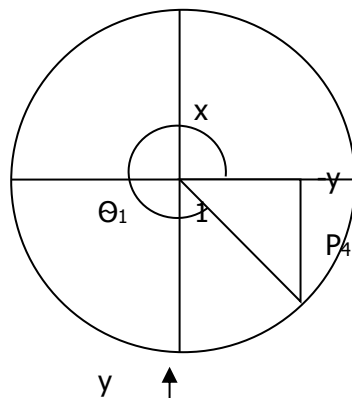
$$: \tan(180^\circ + \theta) = \tan \theta$$

Hence in the third quadrant:

$$\begin{aligned}\sin(180^\circ + \theta) &= -\sin\theta \\ \cos(180^\circ + \theta) &= -\cos\theta \\ \tan(180^\circ + \theta) &= \tan\theta\end{aligned}$$

Fourth Quadrant

(a)



(b)

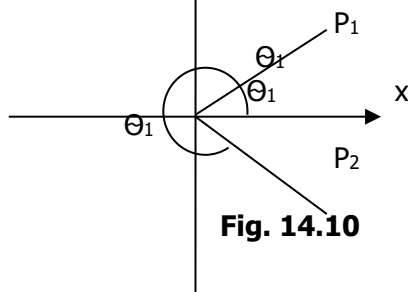


Fig. 14.10

$$\sin \theta_4 = y$$

$$\cos \theta_4 = x$$

$$\tan \theta_4 = \frac{-y}{x}$$

From Fig, 14.10(b)

$$\theta_4 + \theta_1 = 360^\circ$$

$$\theta_4 = 360^\circ - \theta$$

$$\sin \theta_1 = \sin(360^\circ + \theta) = -y = -\sin \theta_1$$

$$: \sin(360^\circ + \theta) = -\sin \theta$$

$$\cos \theta_1 = \cos(360^\circ + \theta) = -x = -\cos \theta_1$$

$$: \cos(360^\circ + \theta) = -\sin \theta$$

$$\tan \theta_4 = \tan(360^\circ + \theta) = \frac{-y}{-x} = -\tan \theta$$

$$: \tan(360^\circ + \theta) = -\tan \theta$$

Hence in the third quadrant:

$$\sin(360^\circ + \theta) = -\sin \theta$$

$$\cos(360^\circ + \theta) = \cos \theta$$

$$\tan(360^\circ + \theta) = -\tan \theta$$

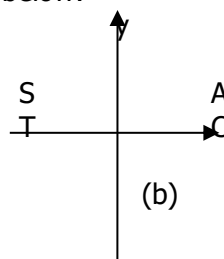
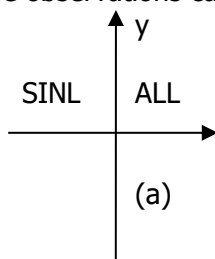
(a) In the first quadrant, all the ratios are positive.

(b) In the second quadrant, only sin ratio is positive, while the rest are negative.

(c) In the third quadrant, only tangent ratio is positive, while the rest are negative.

(d) In the fourth quadrant, only cosine ratio is positive, while the rest are negative.

These observations can be summarized in the figure below:



Negative Angles

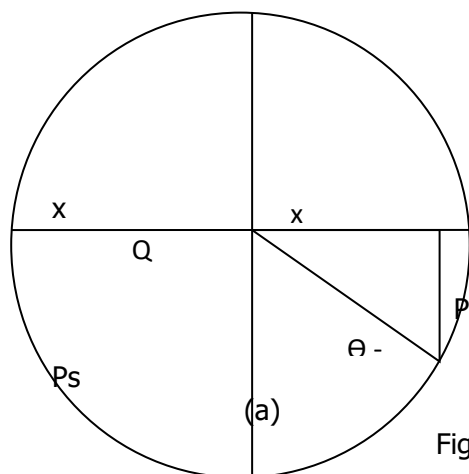
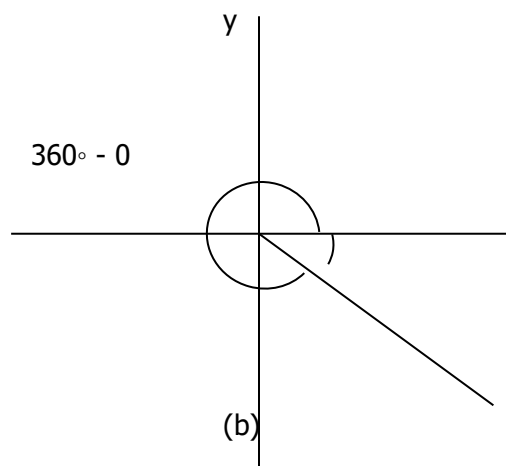


Fig. 14.12



Since negative angles are measured in the clockwise sense, the direction of OP when rotated through $-\theta$ is the same as where it is rotated through $360^\circ - \theta$.

Hence in the fourth quadrant:

$$\sin(-\theta) = \sin(360^\circ - \theta) = -\sin\theta$$

$$\cos(-\theta) = \cos(360^\circ - \theta) = \cos\theta$$

$$\tan(-\theta) = \tan(360^\circ - \theta) = -\tan\theta$$

Use tables to evaluate each of the following:

(a) $\sin 143^\circ$

(b) $\cos 115^\circ$

(c) $\tan 125^\circ$

Solution

(a) 143° is in the second quadrant, so

$$\sin 143^\circ = \sin(180^\circ - 143^\circ)$$

$$= \sin 37^\circ$$

$$= 0.6018$$

(b) 115° is in the second quadrant, so

$$\cos 115^\circ = -\cos(180^\circ - 115^\circ)$$

$$= -\cos 65^\circ$$

$$= -0.4226$$

(c) 125° is in the second quadrant, so

$$\tan 125^\circ = -\tan(180^\circ - 125^\circ)$$

$$= -\tan 55^\circ$$

$$= -1.428$$

Use tables to evaluate each of the following

(a) $\sin 230^\circ$

(b) $\cos 236^\circ$

(c) $\tan 242^\circ$

Solution

220° , 236° and 242° are all in the third quadrant, hence;

(a) $\sin 220^\circ = \sin(180^\circ + 40^\circ)$

$$= -\sin 40^\circ$$

$$= -0.6428$$

$$\begin{aligned} \text{(b) } \cos 236^\circ &= \cos(180^\circ + 56^\circ) \\ &= -\cos 56^\circ \\ &= -0.5992 \end{aligned}$$

$$\begin{aligned} \text{(c) } \tan 242^\circ &= \tan(180^\circ + 62^\circ) \\ &= \tan 62^\circ \\ &= 1.881 \end{aligned}$$

Use tables to evaluate each of the following:

$$\begin{array}{ll} \text{(a) } \sin 310^\circ & \text{(b) } \cos 285^\circ \\ \text{(c) } 334^\circ & \end{array}$$

Solution

310° , 285° and 334° are all in the fourth quadrant, hence;

$$\begin{aligned} \text{(a) } \sin 310^\circ &= \sin(360^\circ - 50^\circ) \\ &= -\sin 50^\circ \\ &= -0.7660 \end{aligned}$$

$$\begin{aligned} \text{(b) } \cos 285^\circ &= \cos(360^\circ - 75^\circ) \\ &= \cos 75^\circ \\ &= 0.2588 \end{aligned}$$

$$\begin{aligned} \text{(c) } \tan 334^\circ &= \tan(360^\circ - 26^\circ) \\ &= -\tan 26^\circ \\ &= -0.4877 \end{aligned}$$

Use tables to evaluate each of the following

$$\begin{array}{ll} \text{(a) } \cos(-30^\circ) & \text{(b) } \sin(-60^\circ) \\ \text{(c) } \tan(-120^\circ) & \end{array}$$

Solution

$$\begin{aligned} \text{(a) } \cos(-30^\circ) &= \cos 330^\circ \\ &= \cos 30^\circ \\ &= 0.8660 \end{aligned}$$

$$\begin{aligned} \text{(b) } \sin(-60^\circ) &= \sin 300^\circ \\ &= -\sin 60^\circ \\ &= -0.8660 \end{aligned}$$

$$\begin{aligned} \text{(c) } \tan(-120^\circ) &= \tan 240^\circ \\ &= \tan 60^\circ \\ &= 1.732 \end{aligned}$$

Use the table to find the value of θ between 0° and 360° which satisfy each of the following:

$$\begin{array}{l} \text{(a) } \cos \theta = -0.4540 \\ \text{(b) } \tan \theta = 1.176 \\ \text{(c) } \sin \theta = -0.9336 \end{array}$$

Solution

(a) The cosine ratio is negative in the second and third quadrants. First find the acute angle whose cosine is 0.4540

From the tables $\cos 63^\circ = 0.4540$

: In the second quadrant

$$\begin{aligned} \theta &= 180^\circ - 63^\circ \\ &= 117^\circ \end{aligned}$$

In the third quadrant,

$$\theta = 180^\circ + 63^\circ$$

$$= 243^\circ$$

(b) The tangent ratio is positive in the first and third quadrants.

First find the acute angle whose tangent is 1.176.

From the tables.

$$\tan 49.62^\circ = 1.176^\circ$$

In the first quadrant.

$$\theta = 49.62^\circ$$

In the third quadrant.

$$\theta = 180^\circ + 49.62^\circ$$

$$= 229.62^\circ$$

(c) The sine ratio is negative in the third and fourth quadrant.

First find the acute angles whose sine ratio is 0.9336.

From tables.

$$\sin 69^\circ = 0.9336$$

In the third quadrant

$$\theta = 180^\circ + 69^\circ$$

$$= 249^\circ$$

In the fourth quadrant.

$$\theta = 360^\circ - 69^\circ$$

$$= 291^\circ$$

Evaluation

1. In what quadrant are the followings ; $\tan (-540)$, $\cos (-1080)$

General Evaluation

(1) Prove that $\frac{(1 - \sin \theta)(1 + \sin \theta)}{\sin^2 \theta} = \cot^2 \theta$

(2) Show that $(\sec \theta - \tan \theta)(\sec \theta + \tan \theta) = 1$

(3) Find the values of $\sin (-210)$ in surd form

Reading Assignment

F/Maths Project 1 pages 225 – 247 Exercise 14 Q1, 3, 5, 7 and 8

Weekend Assignment

Given that $\sin \theta = \frac{4}{5}$ and θ is acute

(1) Find $\cos \theta$ (a) $\frac{5}{3}$ (b) $\frac{3}{5}$ (c) $\frac{4}{3}$ (d) $\frac{4}{5}$

(2) Find $\tan \theta$ (a) $\frac{4}{5}$ (b) $\frac{3}{5}$ (c) $\frac{5}{4}$ (d) $\frac{3}{4}$

(3) Find $\operatorname{cosec} \theta$ (a) $\frac{4}{5}$ (b) $\frac{3}{5}$ (c) $\frac{5}{4}$ (d) $\frac{3}{4}$

(4) Find $\sec \theta$ (a) $\frac{5}{3}$ (b) $\frac{5}{4}$ (c) $\frac{3}{4}$ (d) $\frac{5}{2}$

(5) Find $\cot \theta$ (a) $\frac{3}{5}$ (b) $\frac{4}{5}$ (c) $\frac{5}{4}$ (d) $\frac{5}{3}$

THEORY

1) Prove that $\frac{1}{1+\cos x} + \frac{1}{1-\cos x} = 2 \operatorname{cosec}^2 x$

2) Given that $\sin x = \frac{5}{13}$ and x is acute find $\operatorname{cosec} x$, $\cot x$ and $\sec x$

WEEK 9

TOPIC: Graphs of Trigonometric Function

The graph of the followings will be considered

(a) $y = \sin \theta$, $0^\circ \leq \theta \leq 360^\circ$

(b) $y = \cos \theta$, $0^\circ \leq \theta \leq 360^\circ$

(c) $y = \tan \theta$, $0^\circ \leq \theta \leq 360^\circ$

The graph of $y = \sin \theta$, $0^\circ \leq \theta \leq 360^\circ$

On the graph sheet, draw a long horizontal axis in the middle. Mark a point O' , 3cm to the left of the origin O . With centre O' draw a circle of radius 2cm. Draw a vertical axis through O . Call the horizontal axis θ – axis and the vertical axis y – axis.

On the θ – axis choose a scale of 2cm to 1 unit. Using your protractor, mark the angles: 0° , 30° , 60° , 90° , 120° , 150° ... 330° as shown in

Draw a horizontal line through 30° on the circle. Draw a vertical line through 30° on the circle. Draw a vertical line through 30° on the θ – axis to meet the horizontal line. Mark the point of intersection of these two lines with a small neat cross. Repeat the above procedure for the angles 60° , 90° , 120° , ... 330° . You will obtain a series of points. Join the points by a smooth curve. The curve you obtain is the graph of $y = \sin \theta$.

Essential features of the graph of $y = \sin \theta$:

(a) The graph of $y = \sin \theta$ forms a wave – like pattern. It is said to **oscillate**.

(b) The maximum value of $y = \sin \theta$ is 1 and it occurs when $\theta = 90^\circ$.

(c) The minimum value of $y = \sin \theta$ is -1 it occurs when $\theta = 270^\circ$.

(d) The graph repeats itself at intervals of 360° . The sine function is an example of a periodic function because it repeats itself at intervals of 360° . The function is said to have a **periodicity** of 360° .

(e) The length AE on the graph is called the **amplitude** of the function.

The Graph of $y = \cos \theta$, $0^\circ \leq \theta \leq 360^\circ$

The graph of $y = \cos \theta$ can be drawn in a manner similar to that of $y = \sin \theta$ except that the angles are measured from **OR** in the clockwise sense as shown in Fig. 14.17. This is so because $\cos \theta = \sin(90^\circ - \theta)$.

Essential features of the graph of $y = \cos \theta$:

(a) All the essential features that hold for the graph of $y = \sin \theta$ also hold for the graph of $y = \cos \theta$.

(b) In addition, the cosine curve lags behind the sine curve by a difference of 90° . The difference is usually called a **Phase difference**. In other words, the cosine curve lags behind the sine curve by a phase difference of 90° .

(c) Both the sine curve and the cosine curve demonstrate some physical phenomena like tidal waves, sound waves alternating currents, e.t.c.

The Graph of $y = \tan \theta$, $0^\circ \leq \theta \leq 360^\circ$

The graph of $y = \tan \theta$ is easier to draw using a table of values than using projections from a unit circle. Make a table of values of $y = \tan \theta$ from 0° to 360° as shown in Table 14.2

Essential features of the tangent curve:

(a) The curve consists of three parts between 0° and 360° .

(b) Since the tangent function is not defined at 90° and 270° , the function is said to be

discontinuous at these points. (c) The curve rises and falls rapidly at angles very close to 90° and 270° respectively. The curve approaches the vertical lines at 90° and 270° but never touches them. These vertical lines at 90° and 270° are called **Asymptotes**. The asymptotes are shown by dotted lines.

(d) The tangent function is also a periodic function. It has a periodicity of 180° .

0	0	30°	60°	75°	105	120	150	180	210	240	255	285	315	330	300	360
°	°				°	°	°	°	°	°	°	°	°	°	°	°

y =	0	0.5	1.7	3.7	-	-	-	0	0.5	1.7	3.7	-	-1	0.5	-	0
tan		8	3	3	3.7	1.7	0.5		8	3	3	3.7		8	1.7	
θ					3	3	8					3			3	

Take a scale of 1cm to represent 30° on the θ-axis and 1cm to represent 1 unit on the y-axis.

Example

Using the same axis, a scale of 1cm to represent 30° on the θ-axis and 2cm to represent 1 unit on the y-axis, draw the graphs of the following relations.

(a) $y = \sin \theta$

(b) $y = 2\sin \theta$

(c) $y = \frac{1}{2} \sin \theta$ in the interval $0^\circ \leq \theta \leq 360^\circ$.

Solution

(Refer to table 14.3 and Fig. 14.19)

We observe that the curves:

$y = \sin \theta$

$y = 2\sin \theta$

$y = \frac{1}{2} \sin \theta$

have the same periodicity (360°), but differ in amplitudes.

The amplitude of $y = \sin \theta$ is 1.

The amplitude of $y = 2\sin \theta$ is 2.

The amplitude of $y = \frac{1}{2} \sin \theta$ is $\frac{1}{2}$.

In general, the curve $y = A\sin \theta$ has amplitude $|A|$ and a periodicity of 360°. This property of the same curve is also a characteristic of the cosine curve.

Evaluation

1. Prove that $\sec^2 \theta + \operatorname{cosec}^2 \theta = (\tan \theta + \cot \theta)^2$.

General Evaluation

(1) Draw the graph of $y = 2\cos x - 1$ in the range $0^\circ \leq x \leq 360^\circ$ at intervals of 30°.

(2) Draw the graph of $y = 3\sin x - 1$ in the range of $0^\circ \leq x \leq 360^\circ$ at intervals of 30°.

(3) Sketch the graph of:

(i) $y = \sin 2x$

(ii) $y = \cos x$

(iii) $y = \sec x$

(iv) $\operatorname{cosec} x$

all at intervals of 30° range $0 \leq x \leq 360$.

Weekend Assignment

Given that $4\cos x + 3\sin x = 5$, find the value of

- | | | | | |
|-----------------------|-------------------|-------------------|--------------------|--------------------|
| (1) $\sin x$ | (a) $\frac{4}{5}$ | (b) $\frac{5}{3}$ | (c) $\frac{3}{5}$ | (d) $\frac{4}{3}$ |
| (2) $\cos x$ | (a) $\frac{4}{5}$ | (b) $\frac{5}{3}$ | (c) $\frac{3}{5}$ | (d) $\frac{4}{3}$ |
| (3) $\tan \theta$ | (a) $\frac{4}{5}$ | (b) $\frac{5}{3}$ | (c) $\frac{3}{5}$ | (d) $\frac{4}{3}$ |
| (4) $\cot \theta$ | (a) $\frac{4}{5}$ | (b) $\frac{5}{3}$ | (c) $\frac{3}{5}$ | (d) $\frac{4}{3}$ |
| (5) $\sin x + \cos x$ | (a) $\frac{2}{4}$ | (b) $\frac{7}{5}$ | (c) $\frac{7}{20}$ | (d) $\frac{7}{12}$ |

Theory

(1) Draw the graph of inverse trig function for $\sin x - \frac{\pi}{2} \leq x \leq \frac{\pi}{2}$

(2) Find the inverse of the following and their domains

(a) $y = \sin x$

(b) $y = \cos x$

(c) $y = \tan x$

(d) $y = \operatorname{cosec} x$

(e) $y = \sec x$

(f) $y = \cot x$

WEEK 10

TOPIC: Trigonometric Identities and graphs of inverse trigonometric ratios Pythagoras Theorem

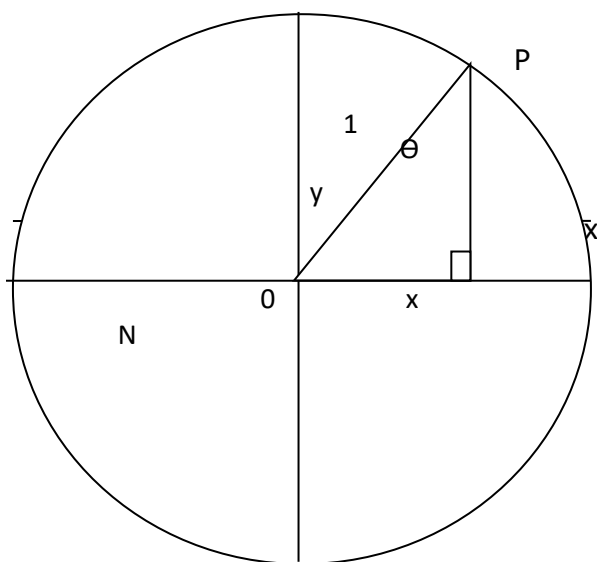


Fig 10a shows a unit circle $\triangle OPN$ is a right – angled triangle with $OP = 1$, $ON = x$ and $PN = y$, $\angle PON = \theta$. Find the definition of trigonometric ratios.

$$x = \cos \theta \quad \dots(1)$$

$$y = \sin \theta \quad \dots(2)$$

$$\text{From (1)} \quad x^2 = \cos^2 \theta \quad \dots(3)$$

$$\text{From (2)} \quad y^2 = \sin^2 \theta \quad \dots(4)$$

Adding (3) and (4)

$$x^2 + y^2 = \cos^2 \theta + \sin^2 \theta \quad \dots(5)$$

Since $\triangle OPN$ is a right – angled triangle

$$ON^2 + NP^2 = OP^2$$

$$x^2 + y^2 = 1 \quad \dots(6)$$

Equating (5) and (6)

$$\cos^2 \theta + \sin^2 \theta = 1 \quad \dots(7)$$

Dividing both sides of (7) by $\cos^2 \theta$

$$\frac{\cos^2 \theta}{\cos^2 \theta} + \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$$

$$: 1 + \tan^2 \theta = \sec^2 \theta$$

Dividing (7) through by $\sin^2 \theta$

$$\cos^2 \theta + 1 = \csc^2 \theta$$

$$: 1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$$

Example

Show that $(3 - \sin^2 \theta) \operatorname{cosec}^2 \theta = 2 \operatorname{cosec}^2 \theta + \cot^2 \theta$

Solution

$$\begin{aligned} \text{LHS} &= 3 \operatorname{cosec}^2 \theta - \sin^2 \theta \operatorname{cosec}^2 \theta \\ &= 3 \operatorname{cosec}^2 \theta - \frac{\sin^2 \theta}{\sin^2 \theta} \end{aligned}$$

$$\begin{aligned}
&= 3\operatorname{cosec}^2\theta - 1 \\
&= 3\operatorname{cosec}^2\theta - (\operatorname{cosec}^2\theta - \cot^2\theta) \\
&= 3\operatorname{cosec}^2\theta - \operatorname{cosec}^2\theta + \cot^2\theta \\
&= 2\operatorname{cosec}^2\theta + \cot^2\theta \\
&= \text{RHS}
\end{aligned}$$

Example

Prove that $\frac{1}{1+\cos\theta} + \frac{1}{1-\cos\theta} = 2\operatorname{cosec}^2\theta$

Solution

$$\begin{aligned}
\text{LHS} &= \frac{(1-\cos\theta)+(1+\cos\theta)}{(1+\cos\theta)(1-\cos\theta)} \\
&= \frac{2}{1-\cos^2\theta}
\end{aligned}$$

$$\begin{aligned}
\text{LHS} &= \frac{2}{\sin^2\theta} \\
&= 2\operatorname{cosec}^2\theta \\
&= \text{RHS}
\end{aligned}$$

Example

Show that $\frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta} = 4\cot^2\theta + 2$

Solution

$$\begin{aligned}
\text{LHS} &= \frac{1+\cos\theta}{1-\cos\theta} + \frac{1-\cos\theta}{1+\cos\theta} \\
&= \frac{(1+\cos\theta)^2 + (1-\cos\theta)^2}{(1-\cos\theta)(1+\cos\theta)} \\
&= \frac{1+2\cos\theta+\cos^2\theta+1-2\cos\theta+\cos^2\theta}{1-\cos^2\theta} \\
&= \frac{2\cos^2\theta+2}{\sin^2\theta} \\
&= \frac{2(\cos^2\theta+1)}{\sin^2\theta} \\
&= 2(\cot^2\theta+\operatorname{cosec}^2\theta) \\
&= 2(\cot^2\theta+1+\cot^2\theta) \\
&= 2(2\cot^2\theta+1) \\
&= 4\cot^2\theta+2 \\
&= \text{RHS}
\end{aligned}$$

Example

If $a\sin\theta + b\cos\theta = p$
and $a\cos\theta - b\sin\theta = q$
show that $a^2 + b^2 = p^2 + q^2$

Solution

$$a\sin\theta + b\cos\theta = p \quad \dots(1)$$

$$a\cos\theta - b\sin\theta = q \quad \dots(2)$$

Squaring both sides of (1)

$$(a\sin\theta + b\cos\theta)^2 = p^2$$

$$: a^2 \sin^2\theta + 2ab\sin\theta \cos\theta + b^2 \cos^2\theta = p^2 \quad \dots(3)$$

Squaring both sides of (2)

$$(a\cos\theta - b\sin\theta)^2 = q^2$$

$$a^2 \cos^2\theta - 2ab\sin\theta \cos\theta + b^2 \cos^2\theta = q^2 \quad \dots(4)$$

Adding (3) and (4)

$$a^2 \sin^2 \theta + 2ab \sin \theta \cos \theta + b^2 \cos^2 \theta + a^2 \cos^2 \theta - 2ab \sin \theta \cos \theta + b^2 \sin^2 \theta = p^2 + q^2$$

$$: a^2 \sin^2 \theta + a^2 \cos^2 \theta + b^2 \cos^2 \theta + b^2 \sin^2 \theta = p^2 + q^2$$

$$a^2 (\sin^2 \theta + \cos^2 \theta) + b^2 (\cos^2 \theta + \sin^2 \theta) = p^2 + q^2$$

$$\text{But } \sin^2 \theta + \cos^2 \theta = 1$$

$$: a^2 + b^2 = p^2 + q^2$$

Trigonometric Equations

A trigonometric equation is an equation containing trigonometric ratios.

Since trigonometric functions are periodic functions, there are infinite numbers of solutions for trigonometric equations. It is therefore the tradition to find some solutions within a given range.

If $a \sin \theta + b = 0$ ($a \neq 0$)

then, $a \sin \theta = -b$

$$\sin \theta = \frac{-b}{a}$$

$$\theta = \sin^{-1} \left[\frac{-b}{a} \right]$$

$\sin^{-1} \left[\frac{-b}{a} \right]$ means the angles whose sine ratio is $\frac{-b}{a}$

It is called sine inverse of $\left[\frac{-b}{a} \right]$ it is also denoted as $\sin^{-1} \left[\frac{-b}{a} \right]$ thus:

$$\sin \theta = \frac{-b}{a}$$

$$\theta = \sin^{-1} \left[\frac{-b}{a} \right]$$

$$= \arcsin \left[\frac{-b}{a} \right]$$

Similarly, if $a \cos \theta + b = 0$, ($a \neq 0$)

$$\text{then } \cos \theta = \frac{-b}{a}$$

$$\theta = \cos^{-1} \left[\frac{-b}{a} \right]$$

Evaluation

Sketch the graph of:

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